EXTENSIONS OF GRAPH LABELING

THESIS

Submitted by

SHERRY GEORGE

(Reg. No. 11728)

MATHEMATICS

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

MANOMANIAM SUNDARANAR UNIVERSITY

TIRUNELVELI - 627 012.

NOVEMBER - 2017

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CERTIFICATE

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SHERRY GEORGE
RESEARCH SCHOLAR

DR. A. LOURDUSAMY
SUPERVISOR
Associate Professor
Department of Mathematics
St. Xavier's College (Autonomous)
Palayamkottai 627 002
Tamil Nadu

Abstract

This thesis is presented as a result of the study done by the author under the guidance of Dr.A.Lourdusamy for the award of the Degree of Doctor in Philosophy in Mathematics from Manonmaniam Sundaranar University, Tirunelveli, Tamilnadu, India.

Graph labeling has developed into one of the important areas in Graph Theory for last fifty years. Though graph labeling is considered primarily a theoretical subject in graph theory and discrete Mathematics, it serves as models for variety of applications. It is used in many applications like coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network, transport problems, data base management etc. to list a few[15]. For each type of application, depending on the problem situation, a type of graph is used for representing the situation. Then a suitable labeling method is applied on that graph and the problem is solved with ease and comfort.

Graph labeling techniques derive its origin to a function named $\beta-$ valuation by Rosa [28] in 1967. He called a function f a $\beta-$ valuation of a graph G with p vertices and q edges if f is an injection from the vertices of G to the set $\{1, 2, \cdots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. He introduced $\beta-$ valuation as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs [7]. Several years later, Golomb [9] studied the same and named it graceful labeling and this name is well known in Graph theory today.

A labeling of a graph G is an assignment of labels either to the vertices or edges. If the domain is the set of vertices, then the labeling is known as vertex labeling. A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge uv depending on the vertex labels. Otherwise it is edge labeling. An edge labeling of

a graph G is assignment f of labels to the edges of G that induces a label for each vertex v depending on the edge labels. There are varieties of vertex as well as edge labeling that are already in the literature [7].

After its origin from $\beta-$ valuation, graph labeling methods grew far and wide. $\beta-$ valuation began as a means of attacking the conjecture of Ringel that K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges [29]. In 1980, Graham and Solane [10] introduced Harmonious labeling which came from their study of modular versions of addittive base problems that arose from error-correcting codes. They defined a graph with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y)(mod\, q)$, the resulting edge labels are distinct.

Later Acharya introduced Super graceful labeling. A $(p, q)-$ graph G is said to be a super graceful graph if there is a bijective function $f : V(G) \to \{1, 2, \cdots, p + q\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ Acharya and Germina [1] further introduced an edge analogue of graceful labeling and named it as vertex graceful numbering. Singh and Devraj [31] brought in the concept of triangular graceful graphs. They call a graph G with p vertices and q edges triangular graceful if there is an injection f from $V(G)$ to $\{1, \cdots, T_q\}$, where T_q is the q^{th} triangular number and the labels induced on each edge uv by $|f(u) - f(v)|$ are the first q triangular numbers.

This way there came into existence many variations of Graceful as well as Harmonious labelings. One significant example is Cordial Labeling introduced by Cahit [6]. For a graph G if the funtion f is from $V(G)$ to $\{0, 1\}$ and for each edge xy the label $|f(x) - f(y)|$ is assigned, then f is called Cordial labeling of G, when the number of vertices labeled 0 and the number of vertices labeled 1 differ at most by 1 and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. The other famous labeling methods are Felicitous labeling, Magic labeling, Antimagic labeling, Mean labeling, Prime labeling etc. The fast growth of this area of study is evident from the fact that more than 2000 papers on graph labeling methods have come out over the past five decades [7].

The concept of Mean labeling was introduced by Somasundaram and Ponraj [32]. A graph G with p vertices and q edges is called a mean graph if there is an injective function f from $V(G)$ to the set $\{0, 1, 2, \cdots, q\}$ that induces for each edge uv the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ such that the set of edge labels is $\{1, 2, \cdots, q\}$. For example, Lourdusamy and Seenivasan [17] have proved that kC_n snakes are mean graphs. Many others have also worked on this notion of graph labeling.

Ramya, Ponraj and Jeyanthi [27] introduced a new variation of mean labeling and named it Super mean labeling. A super mean labeling f is an injection from V to the set $\{1, 2, \cdots, p+q\}$ that induces for each edge uv the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ such that the set of all vertex labels and the induced edge labels is $\{1, 2, \cdots, p+q\}$. They have proved that many graphs, like paths, combs, odd cycles, P_n^2 etc. are super mean graphs. Jeyanthi, Ramya and Thangavelu in [12] have proved that graphs like $nK_{1,4}$ are super mean graphs. Again they in [13] proved that the graph obtained by identifying endpoints of two or more copies of P_5; the graph obtained from C_n by joining two vertices of C_n distance 2 apart with a path of length of two or three etc. are super mean graphs. In [14] Jeyanthi, Ramya and Thangavelu give super mean labelings for $C_m \cup C_n$ and $k-$ super mean labelings for many graphs.

Balaji, Ramesh and Subramanian use in [2] and [3] *Skolem mean labeling* for super mean labeling. They too have proved a variety of graphs to be Skolem mean graphs. Nagarajan, Vasuki and Arockiaraj [24] introduced the concept of Super Mean Number of a graph. They were inspired by [33] Sundaram, Ponraj etc., who brought in the concept of *Mean Number*. Let G be a graph and let $f : V(G) \to \{1, 2, \cdots, n\}$ be a function such that the label of the edge uv is $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ and $f(V(G)) \cup f^*(E(G)) \subseteq \{1, 2, \cdots, n\}$. If n is the smallest positive integer satisfying these conditions together with the condition that all the vertex and edge labels are distinct and there is no common vertex and edge labels, then n is called the super mean number of the graph G and is denoted by $S_m(G)$. They also have proved in [24] that for any graph G of order p, $S_m(G) \leq 2^P - 2$ and have provided an upper bound of super mean number of a few graphs. Some results on mean labeling and super mean labeling are given in [12], [13], [14], [15], [17], [25], [30], [34] etc.

Gayathri and Tamilselvi in [7] brought the notion of $(k, d)-$ super mean labeling defined as follows; A (p, q)- graph G has a $(k, d)-$ super mean labeling if there exists an injection f

from the vertices of G to $\{k, k+1, \cdots, k+(p+q)d\}$ such that the induced map f^* defined on the edges of G by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ has the property that the vertex labels and edge labels together are the integers from k to $k+(p+q)d$. When $d = 1$, a $(k, d)-$super mean labeling is called a $k-$super mean labeling. In [14] the authors enlist many $k-$super mean graphs.

Lourdusamy and Seenivasan [16] introduced vertex mean labeling as an edge analogue of mean labeling as follows: A vertex mean labeling of a (p, q) - graph $G(V, E)$ is defined as an injection $f : E \longrightarrow \{0, 1, \cdots, q*\}$, $q*$ = max(p, q) such that the injection $f : V \to N$ defined by the rule $f^v(V)$ = Round$\left(\frac{\sum_{e\in E_v} f(e)}{d(v)}\right)$ satisfies the property that $f^v(V) = \{f^v(u) : u \in V\} = \{1, 2, ..., p\}$, where E_v denotes the set of edges in G that are incident at v and N denotes the set of all natural numbers. A graph that has a vertex mean labeling is called a vertex mean graph or $V-$mean graph. They have obtained necessary conditions for a graph to be a vertex mean graph and have named a number of vertex mean graphs in [30].

Inspired and motivated by above developments in graph labeling we introduce another variation of mean labeling, named *Super Vertex Mean Labeling*. This type of labeling is a variation of both Super mean labeling and Vertex mean labeling. A Super Vertex Mean labeling f of a (p, q) - graph $G(V, E)$ is defined as an injection from E to the set $\{1, 2, 3, \cdots, p+q\}$ that induces for each vertex v the label defined by the rule $f^v(v)$ = Round $\left(\frac{\sum_{e\in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v, such that the set of all edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, p+q\}$. A graph that admits such labeling is known as Super Vertex Mean graph (SVM). Super vertex mean graphs can be viewed the dual of Super mean graphs, especially in the case of $2-$regular graphs like cycles, $C_n, (n \geq 3)$. In this sense our study is an extension work of these two concepts; Vertex mean labeling and Super mean labeling. Super vertex mean behaviour of many standard graphs has been studied and recorded in this thesis. Attempt is also made to construct new types of Super Vertex Mean graphs and related concepts pertaining to graph labeling techniques.

The thesis is presented in seven chapters. The first chapter gives a few preliminary concepts in graph theory and in the field of graph labeling methods that are needed in the upcoming chapters. For the terminologies that are not explicitly mentioned here, a humble request is made to refer Bondy and Murty [5], Gary Chartand [8] and West [35].

Chapter 2 introduces the subject matter of the thesis and defines the concept: Super Vertex Mean labeling. As no tree is an Super Vertex Mean (SVM) graph we begin our discussion with cycles $C_n, (n \geq 3)$ and fans $F_n, (n \geq 2)$ and a variety of ways a cycle can be labeled in such a fashion. We also define the concept called *Super Vertex Mean Number* of a graph G, inspired by the similar concepts namely, mean number and super mean number that are already in the literature [33] and [24].

In Chapter 3, we study graphs that admit super vertex mean labeling. Here we present and prove that cyclic snakes, $kC_n, (n \geq 3)$ of a particular category are SVM graphs. Every cyclic snake is represented by a unique string of integers. This typical category of snakes contains strings in which each integer is 1.

Chapter 4 deals with linear cyclic snakes, of which all of them are SVM graphs. Linear cyclic snake is a cyclic snake whose string contains integers each of which is equal to $\lfloor \frac{n}{2} \rfloor$, where n is the order of the individual constituent cycle in the cyclic snake.

In Chapter 5, our investigation continues on a third type of cyclic snake known as edge linked cyclic snake $(EL(kC_n))$. In his Ph.D. thesis, of 2013, Seenivasan [30] has defined edge linked cyclic snake as an edge analogue of kC_n snakes. He has generalized edge linked cyclic snake and analysed the conditions under which they are mean graphs. Here we continue our quest by investigating these graphs in the realm of super vertex mean labeling.

Chapter 6 deals with SVM behaviour of all the graphs up to order 5 and all the regular graphs up to order 7. In doing so we have attempted to prove that disjoint union of SVM graphs is SVM graph. The converse of the above fact is not true as C_4 is not a SVM graph but its union with any cycle, including with itself ($2C_4$), is a SVM graph.

Chapter 7 goes one step ahead of what has been proved in the previous chapter 6 and brings in a new result that disjoint union of any type and number of cycles of any order is a SVM graph. In this chapter a method of labeling such union of graphs is described. Attempts are also made to prove $P_n^2, (n \geq 3)$ and graphs obtained from $C_n, (n \geq 4)$, by joining two vertices of C_n, which are of certain distance apart, with a chord, are SVM graphs.

Acknowledgement

It is pleasure to thank the many who have accompanied me during my time in Palayamkottai and made this thesis possible.

First of all to God the Almighty, without whose blessing, graceful presence, wisdom and guidance anything is impossible, I offer my hearts full of gratitude.

Next I place on record, my deepest feelings of gratitude to my competent and generous supervisor and guide, **Dr.A.Lourdusamy, M.Sc., M.Phil., B.Ed., Ph.D.**, Associate Professor, Department of Mathematics, St.Xavier's College(Autonomous), Palayamkottai for his assistance and valuable discussions over the past three years and for all the time and efforts he has put into this work. I thank him for his meticulous guidance, prompt corrections, constructive criticisms and personal encouragements throughout my time as a Ph.D. student.

I also thank the always approachable Principals of St.Xavier's College and friendly staff at the Department of Mathematics for their constant support in providing the necessary help when required. I gratefully remember their encouragement and caring concern to carry out the research work fruitfully.

Finally I thank all those well wishers of mine, who showed fullest cooperation possible and took keenest interest in my research work.

Place : Palayamkottai

Date : **Sherry George**

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Chapter 1

Preliminaries

In this chapter we list some basic concepts of our area of study, which are needed in the subsequent chapters. Terms and terminologies connected with Graph theory and labeling are discussed at length here. For concepts in graph theory, the reader can refer to [5].

1.1 Terms in Graph Theory

Definition 1.1.1. *A graph G is a triple consisting of a finite non empty set, called the vertex set of G and is denoted by $V(G)$, of objects called vertices (also called points or nodes), a(possibly empty) set called the edge set denoted by $E(G)$ of two element subsets of $V(G)$ called edges(or lines), and an incidence function, ψ_G, that associates with each edge two vertices (not necessarily distinct) called endpoints.*

The number of vertices in a graph G is called its order, and the number of edges is its size. In general, for a graph G we use p to denote its order and q for its size. A graph of order p and size q is called a $(p, q)-$graph.

If e is an edge and u and v are vertices of a graph G such that $\psi_G(e) = uv$, then e is said to join u and v, and the vertices u and v are called the ends of e. When a vertex v is an endpoint of some edge e, we say that e is incident with the vertex v and that v is incident with the edge e. Two vertices u and v of a graph G is said to be adjacent if there exists an edge $e \in E(G)$ such that $\psi_G(e) = uv$. Two edges are said to be adjacent if they have a common end vertex.

If e is an edge from a vertex v to itself, then it is called a loop on the vertex v. The incidence function ψ_G need not be one-one. Therefore, it is possible the $\psi_G(e_1) = \psi_G(e_2)$. Then e_1 and e_2 are called parallel edges. A vertex v of graph G is called an isolated vertex if it is not incident with any edge in G. A graph is called simple if it has no loops and no parallel edges. It is possible that a graph G can have directed edges or arcs. Such a graph is known as directed graph or Digraph. The graphs considered here will be finite, undirected and simple.

Definition 1.1.2. *The degree of a vertex v of G is the number of edges incident on it and is denoted by $d(v)$. A vertex with degree zero is called an isolated vertex; a vertex with degree one is a pendant vertex or a leaf. The unique edge that is incident with a pendant vertex is a pendant edge. A vertex with odd degree is an odd vertex and that with an even degree is an even vertex.*

Throughout this thesis the letter G denotes a graph. Moreover, when there is no scope of ambiguity, the letter G is omitted from graph-theoretic symbols and write, for example V and E instead of $V(G)$ and $E(G)$ respectively.

Definition 1.1.3. *A walk in a graph is finite non empty sequence whose terms are alternatively vertices and edges. If the edges of a walk are distinct, then the walk is called a trail and in addition, if the vertices are distinct then the walk is known as a path. A path with n vertices is denoted by P_n. A walk, trail or path is called trivial if it has only one vertex and no edges.*

Definition 1.1.4. *If in a $u - v$ walk $u = v$ then we say that the walk is closed. A non-trivial closed trail is called a circuit. A non-trivial closed trail in a graph G is called a cycle if its origin and internal vertices are distinct. In detail, the closed trail $C = v_1v_2 \cdots v_nv_1$ is a cycle if C has at least one edge and $v_1, v_2, \cdots, v_n$ are distinct vertices. A cycle with n edges is an $n-$cycle. An $n-$cycle is called odd or even depending on whether n is odd or even* *respectively. An $n-$cycle is commonly denoted by C_n. Every cycle is a circuit, but a circuit need not be a cycle.*

Definition 1.1.5. *Let u and v be two vertices of a graph G. The vertex u is said to be connected to the vertex v if there exists a $u-v$ walk in G. The graph itself is said to be connected if for every two pair u, v of vertices of G there is a $u-v$ walk in G. Otherwise graph G is said to be disconnected.*

Definition 1.1.6. *For a non-trivial graph G and a pair u, v of vertices of G, the distance between u and v is the length of the shortest $u-v$ path in G, if it exists. It is denoted by $d_G(u, v)$. If G has no such $u-v$ path, then we define $d_G(u, v) = \infty$.*

Definition 1.1.7. *A graph that is connected and has no cycles is known as a tree. Every non-trivial tree has at least two pendant vertices.*

Definition 1.1.8. *A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to $G_2 = (V_2, E_2)$ if there is a one-to-one correspondence between the vertex sets V_1 and V_2 and a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1 then corresponding edge e_2 in G_2 has its end vertices u_2 and v_2 in G_2 which correspond to u_1 and v_1 respectively. Such a pair of correspondence is called graph isomorphism.*

Definition 1.1.9. *A complete graph of order n, denoted by K_n is a simple graph in which each pair of distinct vertices is joined by an edge. Thus, a graph with n vertices is complete if it has as many as possible edges, provided there are no loops and no multiple edges.*

Definition 1.1.10. *Two graphs G_1 and G_2 are said to disjoint if they have no vertex in common, and they are edge disjoint if they have no edge in common.*

Definition 1.1.11. *Let G_1 and G_2 be two graphs, the union $G_1 \cup G_2$ is a graph G with vertex set consisting of all those vertices which are either in G_1 or G_2 (or both) and with edge set consisting of all those edges which are either in G_1 or G_2 (or both). The disjoint union of m copies of a graph G is denoted by mG.*

Definition 1.1.12. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. The direct product of G and H, $G \times H$, whose vertex set is the Cartesian product $V(G \times H) = V_G \times V_H = \{(x, y) : x \in V_G, y \in V_H\}$ and whose edges are given by $E_{G\times H} = \{(x, y), (x', y') : x = x'$ and $(y, y') \in E_H$ or $(x, x') \in E_G$ and $y = y'\}$. The product $P_m \times P_n$ is called a planar grid and $P_2 \times P_n$ is known as a ladder. The product $C_m \times P_n$ is called a prism.*

Definition 1.1.13. *The Square of graph G denoted by G^2 has the same vertex set as that of G and the two vertices are in G^2 if they are at a distance of 1 or 2 in G.*

Definition 1.1.14. *If for some positive inter r, $d(v) = r$ for every vertex v of the graph G, then G is called $r-$regular. A 3-regular graph is also called a cubic graph. The complete graph K_n is $(n - 1)$- regular graph. The complete bipartite graph $K_{n,n}$ on $2n$ vertices is n-regular.*

1.2 Labeling and Number Theoretic Terms

Definition 1.2.1. *For non-empty sets A and B, a function f from A to B, written as $f : A \to B$, is a relation from A to B in which each element of A appears as the first coordinate in exactly one ordered pair. If the ordered pair $(a, b) \in f$, then we write $b = f(a)$ and b is the image of a. The set of all images of f is called the range of f.*

Definition 1.2.2. *A function $f : A \to B$, is injective (or one-to-one) if distinct elements of A have distinct elements in B. Therefore, f is injective if for every two(distinct) elements a_1 and a_2 in A, it follows that $f(a_1) \neq f(a_2)$.*

Definition 1.2.3. *A function $f : A \to B$, is surjective (or onto) if every element of B is the image of some element of A, i.e., if the range of f is B.*

Definition 1.2.4. *A function that is both injective and surjective is called a bijective function or a one-to-one correspondence.*

Definition 1.2.5. *A labeling of a graph G is a map that carries graph elements to integers. Or in other words, a labeling of a graph G is a function either from the set of vertices or the* *set of edges to a set of integers such that there is an induced function from the set of edges or the set of vertices respectively depending on the former function. If the domain is the set of vertices, then the labeling is known as vertex labeling, and if the domain is the set of edges, then the labeling is known as edge labeling.*

Definition 1.2.6. *Round of a number or rounding function of a numerical value means replacing it by another value that is approximately equal but has a shorter, simpler or more explicit representation. The round function is also called the nearest integer function and is defined such that Round(x) is the integer closest to x.*

Definition 1.2.7. *The floor and ceiling functions map a real number to the greatest proceeding or the least succeeding integer, respectively. More precisely, $floor(x) = \lfloor x \rfloor$ is the greatest integer less than or equal to x and $ceiling(x) = \lceil x \rceil$ is the least integer greater than or equal to x.*

Chapter 2

Super Vertex Mean Labeling

This chapter begins with the definition of *Super Vertex Mean Labeling*, which is basically an edge labeling of graphs. It can also be seen as a dual of Super Mean Labeling introduced by Ramya et.al in [27]. Therefore Super Mean Labeling is a variation of both Super mean labeling and the one introduced by Lourdusamy et.al in [16], viz., Vertex mean labeling. In this chapter we examine Cycles, $C_n, n \geq 3$ and Fans, $F_n, n \geq 2$.

Definition 2.0.8. *A Super Vertex Mean labeling f of a (p, q) - graph $G(V, E)$ is defined as an injection from E to the set $\{1, 2, 3, \cdots, p+q\}$ that induces for each vertex v the label defined by the rule $f^v(v)$ = Round $\left(\frac{\sum_{e \in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v, such that the set of all edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, p+q\}$.*

A graph that admits super vertex mean labeling is called a Super Vertex Mean, that is, **SVM**) graph in short.

2.1 A Preliminary Observation

A graph having isolated vertices or leaves cannot be an SVM - graph. For, if $deg(v) = 0$ for any vertex v of G, the above definition is not defined and if $deg(v) = 1$ for any vertex v of G, the induced vertex label remains the same as the label of the edge that is incident on the vertex v. Therefore, necessarily $deg(v) \geq 2$ for all vertices v of G. It is obvious that no tree is an SVM - graph.

2.2 Super Vertex Mean Labeling of Cycles

Theorem 2.2.1. *All the cycles except C_4 are SVM - graphs.*

Proof. It is clear from the following illustration that C_4 is not SVM - graph.

Illustration: For C_4, $p = 4$ and $q = 4$.

$f(E) \cup f(V) = \{1, 2, 3, \cdots, p + q\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

It is obvious that 1 and 8 cannot be induced vertex labels, so necessarily belong to $f(E)$. Since 2 cannot be an edge label, it belongs to $f(V)$ and for 2 to be a vertex label, it has to labeled on a vertex on which the edges that are labeled 1 and 3 lie. And so, 3 also belongs to $f(E)$.

Therefore, 8 can be labeled on an edge that is adjacent to an edge labeled 3 or 1. The following cases emerge:

Case 1: Let 8 be labeled on an edge adjacent to the edge labeled 3.

Now, 7 cannot be labeled on any edges. The remaining options are that, we label either 4 or 5 on the fourth edge.

Case 1.a.: Let 4 be labeled on the fourth edge. This is not an SVM - labeling as the vertices that are incident on the edge labeled 8 get the same induced label 6.

Case 1.b.: Let 5 be labeled on the fourth edge. This also is ruled out as one of the vertices incident on the edge labeled 5 gets the label 3, which is contrary to the assumption that 3 has to be an edge label.

Therefore **case 1** is not possible.

Case 2: Let 8 be labeled on an edge which is adjacent to the edge labeled 1.

In this case 7 cannot be an edge label and if 7 were to become an induced vertex label, then one of the induced vertex labels gets repeated. Therefore this **case 2** also is impossible.

The above investigation reveals that the cycle C_4 is not an SVM - graph. So, let us assume that $n \neq 4$.

Now let us prove that C_n, except C_4 is an SVM - graph. There can be two cases depending upon whether n is odd or even.

Case 3: $n \equiv 1(mod\, 2)$.

Let C_n be an odd cycle with n vertices. Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n, such that $e_i = v_i v_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_n v_1$.

Let $n = 2r+1$. The edges of C_n are labeled as follows:

$$f(e_i) = \begin{cases} 2i-1 & \text{if } 1 \leq i \leq r+1 \\ 2i & \text{if } r+2 \leq i \leq n \end{cases}$$

It is easy to observe that f is injective. The induced vertex labels are given as follows:

$$f^v(v_i) = \begin{cases} n+1 & \text{if } i = 1 \\ 2i-2 & \text{if } 2 \leq i \leq r+1 \\ 2i-1 & \text{if } r+2 \leq i \leq n \end{cases}$$

It is clear that,

$$\begin{aligned} f(E) \cup f^v(V) &= \{1, 3, 5, \cdots, 2r+1, 2r+4, 2r+6, \cdots, 2n-2, 2n\} \cup \\ &\{2r+2 = n+1, 2, 4, \cdots, 2r-2, 2r, 2r+3, 2r+5, \cdots, 2n-3, 2n-1\} \\ &= \{1, 3, \cdots, 2r+1 = n, 2r+3, 2r+5, ..., 2n-1\} \cup \\ &\{2, 4, \cdots, 2r = n-1, n+1 = 2r+2, 2r+4, 2r+6, \cdots, 2n-1, 2n\} \\ &= \{2i-1 : 1 \leq i \leq n\} \cup \{2i : 1 \leq i \leq n\} \\ &= \{1, 2, 3, \cdots, 2n\} \end{aligned}$$

Case 4: $n \equiv 0(mod\, 2)$

Let C_n be an even cycle with n vertices. Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_i v_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_n v_1$.

Let $n = 2r$. The edges of C_n are labeled as follows:

$$f(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 3 & \text{if } i = 2 \\ 7 & \text{if } i = 3 \\ 4i - 4 & \text{if } 4 \leq i \leq r + 1 \\ 4n - 4i + 5 & \text{if } r + 2 \leq i \leq n - 1 \\ 6 & \text{if } i = n \end{cases}$$

It is easy to observe that f is injective. The induced vertex labels are given as follows:

$$f^v(v_i) = \begin{cases} 4 & \text{if } i = 1 \\ 2 & \text{if } i = 2 \\ 5 & \text{if } i = 3 \\ 4i - 6 & \text{if } 4 \leq i \leq r + 1 \\ 4n - 4i + 7 & \text{if } r + 2 \leq i \leq n - 1 \\ 8 & \text{if } i = n \end{cases}$$

It is clear that,

$$\begin{aligned} f(E) \cup f^v(V) &= \{1, 3, 7, 12, 16, \cdots, 4r, 4r - 3, 4r - 7, \cdots, 13, 9, 6\} \cup \\ &\quad \{4, 2, 5, 10, 14, \cdots, 4r - 6, 4r - 2, 4r - 1, 4r - 5, \cdots, 15, 11, 8\} \\ &= \{1, 3, 6, 7, 12, 16, 20, \cdots, 4r, 9, 13, \cdots, 4r - 7, 4r - 3\} \cup \\ &\quad \{2, 4, 5, 8, 10, 14, 18, \cdots, 4r - 6, 4r - 2, 11, 15, 19, \cdots, 4r - 5, 4r - 1\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8\} \cup \{9, 13, 4r - 3\} \cup \{10, 14, \cdots, 4r - 2\} \cup \\ &\quad \{11, 15, \cdots, 4r - 1\} \cup \{12, 16, \cdots, 4r\} \\ &= \{1, 2, 3, \cdots, 4r - 3, 4r - 2, 4r - 1, 4r = 2n\} \\ &= \{1, 2, 3, \cdots, 2n\} \end{aligned}$$

Hence we have proved that all Cycles C_n, except C_4 are Super Vertex Mean graphs. □

Example 2.2.2. *Super vertex-mean labeling of C_9 and C_{10} is shown in Figure 2.1.*

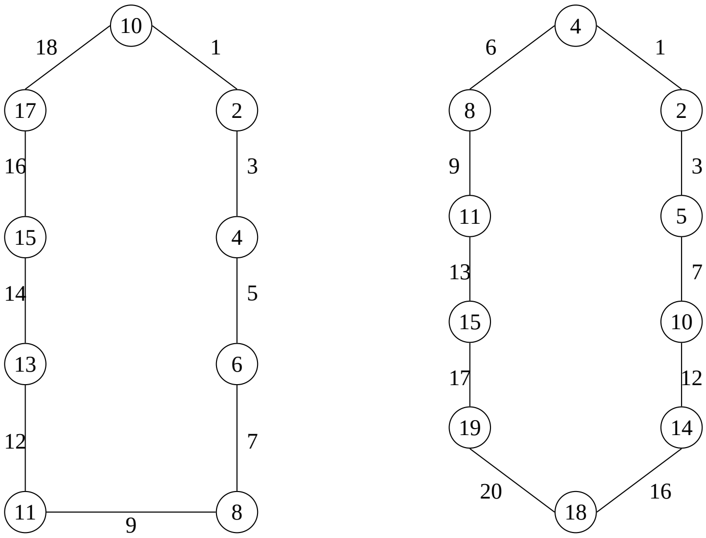


Figure 2.1: Super vertex mean labelings of C_9 and C_{10}.

2.3 Types of SVM - labeling of Cycles

Any cycle C_n, $n \geq 3$ and $n \neq 4$ can be SVM - labeled in a variety of ways. Therefore, the need arises to categorize various types of these labelings.

Example 2.3.1. *Figure* 2.2 *shows that* C_7 *can be labeled altogether as many as* 3 *different ways.*

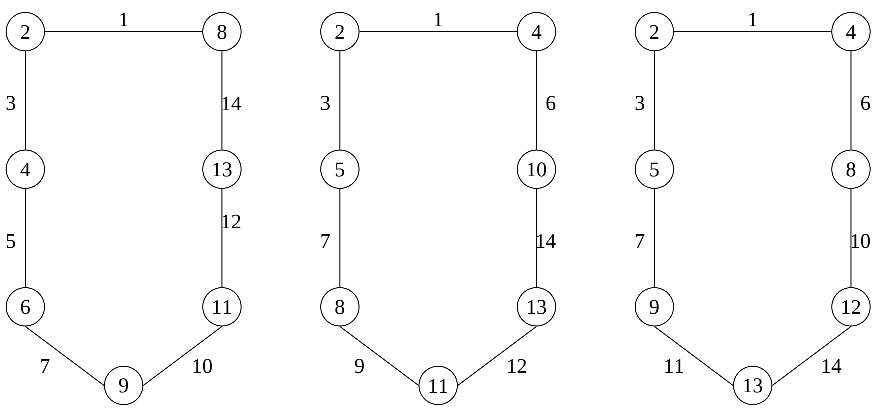


Figure 2.2: C_7 can be labeled altogether as many as 3 different ways.

Definition 2.3.2. ***s – type labeling of Cycles,*** *$C_m, m \geq 3$ and $m \neq 4$: We denote a super vertex mean labeling $f : E \rightarrow \{1, 2, \cdots, 2m\}$ of a cycle C_m, $m \geq 3$ and $m \neq 4$, that places 1 and $2m$ on two edges such that the number of internal vertices along the shortest path connecting these two edges is s, as s-type labeling, where $1 \leq s \leq \left\lfloor \frac{m}{2} \right\rfloor$.*

2.4 Type s – labeling of all cycles

In order to define completely the various types of SVM - labeling of $C_n, n \geq 3$ and $n \neq 4$, we have to consider the following two cases, based on whether n is odd or even.

Case 1: $n \equiv 1 (mod\, 2)$

Let $n = 2r + 1$. Let C_n be an odd cycle with n vertices. Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_i v_{i+1}$, $1 \leq i \leq n - 1$ and $e_n = v_n v_1$. The type 1 - labeling of cycle $C_n, n \geq 3$ is given as follows;

$$f_1(e_i) = \begin{cases} 2i - 1 & \text{if } 1 \leq i \leq r + 1 \\ 2i & \text{if } r + 2 \leq i \leq n \end{cases}$$

or, when we reverse the order of naming the edges and vertices, we get equivalently

$$f_1(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4r - 2i + 6 & \text{if } 2 \leq i \leq r + 1 \\ 4r - 2i + 5 & \text{if } r + 2 \leq i \leq n. \end{cases}$$

Type 2 - labeling, then is defined as follows for $C_n, n \geq 5$;

$$f_2(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } i = 2 \\ 4r + 4 - 2i + 4 & \text{if } 3 \leq i \leq r + 1 \\ 4r + 4 - 2i + 3 & \text{if } r + 2 \leq i \leq 2r \\ 8r - 4i + 7 & \text{if } i = n \end{cases}$$

Similarly type 3 - labeling of $C_n, n \geq 7$ can be defined as,

$$f_3(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } 2 \leq i \leq 3 \\ 4r + 6 - 2i + 4 & \text{if } 4 \leq i \leq r + 1 \\ 4r + 6 - 2i + 3 & \text{if } r + 2 \leq i \leq 2r - 1 \\ 8r - 4i + 7 & \text{if } 2r \leq i \leq n. \end{cases}$$

And when $r = s$, type r - labeling of $C_n, n \geq 3$ is defined as,

$$f_r(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } 2 \leq i \leq r + 1 \\ 8r - 4i + 7 & \text{if } r + 2 \leq i \leq n. \end{cases}$$

or, equivalently

$$f_r(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } 2 \leq i \leq r = s \\ 4r + 2r - 2i + 4 & \text{if } i = r + 1 = s + 1 \\ 4r + 2r - 2i + 3 & \text{if } i = r + 2 = s + 2 \\ 8r - 4i + 7 & \text{if } r + 3 \leq i \leq n. \end{cases}$$

Therefore, when we consider all odd cycles and all the types of their SVM - labeling in general, we have the following theorem.

Theorem 2.4.1. *Let $n = 2r+1$. Let C_n be an odd cycle with n vertices. Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_iv_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_nv_1$. Then type s - $(1 \leq s \leq r)$ SVM - labeling of cycle $C_n, n \equiv 1(mod\, 2), n \geq 3$*

is given as follows:

$$f_s(e_i) = \begin{cases} 1 & \textit{if } i = 1 \\ 4i - 2 & \textit{if } 2 \leq i \leq s \\ 4r + 2s - 2i + 4 & \textit{if } s + 1 \leq i \leq r + 1 \\ 4r + 2s - 2i + 3 & \textit{if } r + 2 \leq i \leq 2r - s + 2 \\ 8r - 4i + 7 & \textit{if } 2r - s + 3 \leq i \leq n. \end{cases}$$

Proof. Let $n \equiv 1(mod\ 2)$, and $n = 2r + 1$. Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_i v_{i+1}, 1 \leq i \leq n - 1$ and $e_n = v_n v_1$. The edges of C_n can type s - labeled, $1 \leq s \leq r$, as given in the theorem. Clearly f_s is an injective function with range $\{1, 2, \cdots, 2n\}$.

The induced vertex labeling is given as follows:

When $s = 1$,

$$f_1^v(v_i) = \begin{cases} 2 & \text{if } i = 1 \\ n + 1 & \text{if } i = 2 \\ 4r - 2i + 7 & \text{if } 3 \leq i \leq r + 2 \\ 4r - 2i + 6 & \text{if } r + 3 \leq i \leq n. \end{cases}$$

It is evident that,

$$\begin{aligned} f_1(E) \cup f_1^v(V) &= \{1, 2n, 2n - 2, 2n - 5, \cdots, n + 5, n + 3, n, n - 2, n - 4, \cdots, 5, 3\} \cup \\ &\quad \{2, n + 1, 2n - 1, 2n - 3, \cdots, n + 4, n + 2, n - 1, n - 3, \cdots, 6, 4\} \\ &= \{1, 3, \cdots, n, n + 3, n + 5, \cdots, 2n\} \cup \\ &\quad \{2, 4, \cdots, n - 1, n + 1, n + 2, n + 4, \cdots, 2n - 1\} \\ &= \{1, 2, 3, \cdots, 2n\}. \end{aligned}$$

When $s = r$,

$$f_r^v(v_i) = \begin{cases} 2 & \text{if } i = 1 \\ 4i - 4 & \text{if } 2 \leq i \leq r + 1 \\ 4r + 1 = 2n - 1 & \text{if } i = r + 2 \\ 8r - 4i + 9 & \text{if } r + 3 \leq i \leq n. \end{cases}$$

It is clear that,

$$\begin{aligned} f_r(E) \cup f_r^v(V) &= \{1, 6, 10, \cdots, 2n, 2n-3, 2n-7, \cdots, 3\} \cup \\ &\quad \{2, 4, 8, \cdots, 2n-2, 2n-1, 2n-5, \cdots, 9, 5\} \\ &= \{1, 3, 7, \cdots, 2n-3, 6, 10, \cdots, 2n\} \cup \\ &\quad \{2, 4, 8, \cdots, 2n-2, 2n-1, 5, 9, \cdots, 2n-5\} \\ &= \{1, 2, 3, 4, 5, \cdots, 2n-3, 2n-2, 2n-1, 2n\}. \end{aligned}$$

Therefore in the more general case, the induced vertex labels are given as follows:

$$f_s^v(v_i) = \begin{cases} 2 & \text{if } i = 1 \\ 4i - 4 & \text{if } 2 \leq i \leq s \\ 2r + 2s & \text{if } i = s + 1 \\ 4r + 2s - 2i + 5 & \text{if } s + 2 \leq i \leq r + 2 \\ 4r + 2s - 2i + 4 & \text{if } r + 3 \leq i \leq 2r - s + 2 \\ 8r - 4i + 9 & \text{if } 2r - s + 3 \leq i \leq n. \end{cases}$$

Clearly it is injective and

$$f_s(E) \cup f_s^v(V) = \{1, 2, 3, 4, 5, \cdots, 2n-3, 2n-2, 2n-1, 2n\}.$$

Since,

$$\begin{aligned} f_s(E) = \{&1, 6, 10, \cdots, 4s-2, 2n, 2n-2, \cdots, 2r+2s+2, 2r+2s-1, \\ &2r+2s-3, \cdots, 4s+1, 4s-1, 4s-5, \cdots, 7, 3\} \\ = \{&1, 3, 6, 7, 10, 11, 14, 15, \cdots, 4s-5, 4s-2, 4s-1, 4s+1, \cdots, \\ &2r+2s-3, 2r+2s-1, 2r+2s+2, \cdots, 2n-2, 2n\} \end{aligned}$$

And,

$$f_s^v(V) = \{2, 4, 8, \cdots, 4s-4, 2r+2s, 2n-1, 2n-3, \cdots, 2r+2s+3, 2r+2s+1,$$
$$2r+2s-2, 2r+2s-4, \cdots, 4s+2, 4s, 4s-3, 4s-7, \cdots, 9, 5\}$$
$$= \{2, 4, 5, 8, 9, \cdots, 4s-4, 4s-3, 2r+2s, 4s, 4s+2, \cdots, 2r+2s-2,$$
$$2r+2s+1, 2r+2s+3, \cdots, 2n-3, 2n-1\}$$
$$= \{2, 4, 5, 8, 9, 12, 13, \cdots, 4s-4, 4s-3, 4s, 4s+2, \cdots, 2r+2s-2,$$
$$2r+2s, 2r+2s+1, 2r+2s+3, \cdots, 2n-3, 2n-1\}$$

Therefore,

$$f_s(E) \cup f_s^v(V) = \{1, 2, 3, 4, 5, \cdots, 2n-3, 2n-2, 2n-1, 2n\}.$$

Hence we have proved that all odd cycles C_n, can be s–type labeled, where $1 \leq s \leq r$ and $n = 2r+1$. □

Case 2: $n \equiv 0 (mod\ 2)$ Let C_n be an even cycle and $n = 2r$ where, $n \geq 6$. (Since C_4 is not an SVM graph).

Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_i v_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_n v_1$.

Checking various possibilities we realize that type 1 - labeling is not possible for even cycles. So we assume that $2 \leq s \leq r$.

Type 2 - labeling of $C_n, n \geq 6$ is given as follows:

$$f_2(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 7 & \text{if } i = 2 \\ 4r - 2i + 6 & \text{if } 3 \leq i \leq r \\ 4r - 2i + 5 & \text{if } r + 1 \leq i \leq 2r - 2 \\ 6 & \text{if } i = 2r - 1 \\ 3 & \text{if } i = 2r. \end{cases}$$

Similarly, type 3 - labeling of C_n, $n \geq 8$ are as follows;

$$f_3(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 7 & \text{if } i = 2 \\ 12 & \text{if } i = 3 \\ 4r - 2i + 8 & \text{if } 4 \leq i \leq r \\ 4r - 2i + 7 & \text{if } r + 1 \leq i \leq 2r - 3 \\ 9 & \text{if } i = 2r - 2 \\ 6 & \text{if } i = 2r - 1 \\ 3 & \text{if } i = 2r \end{cases}$$

And type 4 - labeling of C_n, $n \geq 10$ is given below;

$$f_4(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 7 & \text{if } i = 2 \\ 4i & \text{if } 3 \leq i \leq 4 \\ 4r - 2i + 10 & \text{if } 5 \leq i \leq r \\ 4r - 2i + 9 & \text{if } r + 1 \leq i \leq 2r - 4 \\ 8r - 4i + 1 & \text{if } 2r - 3 \leq i \leq 2r - 2 \\ 6 & \text{if } i = 2r - 1 \\ 3 & \text{if } i = 2r \end{cases}$$

and, when $s = r$, type r - labeling is given by,

$$f_r(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i - 2 & \text{if } 2 \leq i \leq r - 1 \\ r + 3i - 3 & \text{if } r \leq i \leq r + 1 \\ 8r - 4i + 3 & \text{if } r + 2 \leq i \leq n. \end{cases}$$

And in general as in the previous case, for all even cycles, type s - labeling is defined in the following theorem:

Theorem 2.4.2. *Let $n = 2r$. Let C_n be an even cycle with n vertices. Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set of C_n such that $e_i = v_i v_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_n v_1$. Type s - $(2 \leq s \leq r-1)$ SVM - labeling of C_n, $n \equiv 0(mod\ 2), n \geq 6$, is given as follows:*

$$f_s(e_i) = \begin{cases} 1 & \textit{if}\ i = 1 \\ 7 & \textit{if}\ i = 2 \\ 4i & \textit{if}\ 3 \leq i \leq s \\ 4r - 2i + 2s + 2 & \textit{if}\ s+1 \leq i \leq r \\ 4r - 2i + 2s + 1 & \textit{if}\ r+1 \leq i \leq 2r - s \\ 8r - 4i + 1 & \textit{if}\ 2r - s + 1 \leq i \leq 2r - 2 \\ 6r - 3i + 3 & \textit{if}\ 2r - 1 \leq i \leq n \end{cases}$$

and, when $s = r$, type r - labeling is given by,

$$f_r(e_i) = \begin{cases} 1 & \textit{if}\ i = 1 \\ 4i - 2 & \textit{if}\ 2 \leq i \leq r - 1 \\ r + 3i - 3 & \textit{if}\ r \leq i \leq r + 1 \\ 8r - 4i + 3 & \textit{if}\ r + 2 \leq i \leq n. \end{cases}$$

Proof. Let $n \equiv 0(mod\ 2)$, and $n = 2r$. Let $\{e_1, e_2, \cdots, e_n\}$ be the edge set and $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of C_n such that $e_i = v_i v_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_n v_1$.
Case 1: When $2 \leq s \leq r - 1$, the edges of $C_n, n \geq 6$ can be type s - labeled as given

below:

$$f_s(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 7 & \text{if } i = 2 \\ 4i & \text{if } 3 \leq i \leq s \\ 4r - 2i + 2s + 2 & \text{if } s+1 \leq i \leq r \\ 4r - 2i + 2s + 1 & \text{if } r+1 \leq i \leq 2r - s \\ 8r - 4i + 1 & \text{if } 2r - s + 1 \leq i \leq 2r - 2 \\ 6r - 3i + 3 & \text{if } 2r - 1 \leq i \leq n \end{cases}$$

Clearly f_s is an injective function with range $\{1, 2, ..., 2n\}$. The induced vertex labeling is given as follows:

When $s = 2$

$$f_s^v(v_i) = \begin{cases} 2i & \text{if } 1 \leq i \leq 2 \\ 2r + 4 & \text{if } i = 3 \\ 4r - 2i + 2s + 3 & \text{if } 4 \leq i \leq r + 1 \\ 4r - 2i + 2s + 2 & \text{if } r + 2 \leq i \leq 2r - 2 \\ 6r - 3i + 5 & \text{if } 2r - 1 \leq i \leq n \end{cases}$$

And when $s \geq 3$, we have

$$f_s^v(v_i) = \begin{cases} 2i & \text{if } 1 \leq i \leq 2 \\ 2i + 4 & \text{if } i = 3 \\ 4i - 2 & \text{if } 4 \leq i \leq s \\ 2r + 2s & \text{if } i = s + 1 \\ 4r - 2i + 2s + 3 & \text{if } s + 2 \leq i \leq r + 1 \end{cases}$$

$$f_s^v(v_i) = \begin{cases} 4r-2i+2s+2 & \text{if } r+2 \le i \le 2r-s \\ 4s-1 & \text{if } i = 2r-s+1 \\ 8r-4i+3 & \text{if } 2r-s+2 \le i \le 2r-2 \\ 6r-3i+5 & \text{if } 2r-1 \le i \le n \end{cases}$$

Clearly it is an injective function and, it is also evident that, when $s = 2$,

$$\begin{aligned} f_2(E) \cup f_2^v(V) &= \{1, 7, 4r, 4r-2, \cdots, 2r+6, 2r+3, 2r+1, \cdots, 9, 6, 3\} \cup \\ &\quad \{2, 4, 2r+4, 4r-5, 4r-3, \cdots, 2r+5, 2r+2, \cdots, 10, 8, 5\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots, 4r-3, 4r-2, 4r-1, 4r\}. \end{aligned}$$

And for $3 \le s \le r-1$,

$$\begin{aligned} f_s(E) \cup f_s^v(V) &= \{1, 7, 12, \cdots, 4s, 4r, 4r-2, \cdots, 2r+2s+2, 2r+2s-1, \\ &\quad 2r+2s-3, \cdots, 4s+3, 4s+1, 4s-3, 4s-7, \cdots, 9, 6, 3\} \cup \\ &\quad \{2, 4, 10, 14, 18, \cdots, 4s-2, 2r+2s, 4r-1, 4r-3, \cdots, 2r+2s+1, \\ &\quad 2r+2s-2, 2r+2s-4, \cdots, 4s+2, 4s-1, 4s-5, 4s-9, \cdots, 11, 8, 5\} \\ &= \{1, 3, 6, 7, 9, 13, \cdots, 4s-7, 4s-3, 12, 16, \cdots, 4s, 4r, 4r-2, \cdots, \\ &\quad 2r+2s+2, 2r+2s-1, 2r+2s-3, \cdots, 4s+3, 4s+1\} \cup \\ &\quad \{2, 4, 5, 8, 10, 14, \cdots, 4s-2, 11, 15, \cdots, 4s-9, 4s-5, 4s-1, 4s+2, \\ &\quad 4s, \cdots, 2r+2s-4, 2r+2s, 4r-1, 4r-3, \cdots, 2r+2s+1\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \cdots, 4r-2, 4r-1, 4r\}. \end{aligned}$$

Case 2: When $s = r$, the edges of C_n, $n \ge 6$ can be type r - labeled as given below:

$$f_r(e_i) = \begin{cases} 1 & \text{if } i = 1 \\ 4i-2 & \text{if } 2 \le i \le r-1 \\ r+3i-3 & \text{if } r \le i \le r+1 \\ 8r-4i+3 & \text{if } r+2 \le i \le n. \end{cases}$$

Clearly, f_r is an injective function with range $\{1, 2, 3, \cdots, 2n\}$. The induced vertex labeling is given as follows:

$$f_r^v(v_i) = \begin{cases} 2i & \text{if } 1 \leq i \leq 2 \\ 4i-4 & \text{if } 3 \leq i \leq r \\ 4r-1 & \text{if } i = r+1 \\ 4r-2 & \text{if } i = r+2 \\ 8r-4i+5 & \text{if } r+3 \leq i \leq n. \end{cases}$$

It is clear now that,

$$f_r(E) \cup f_r^v(V) = \{1, 2, 3, \cdots, 2n-2, 2n-1, 2n\}.$$

Since,

$$f_r(E) = \{1, 6, 10, \cdots, 4r-6, 4r-3, 4r, 4r-5, 4r-9, 4r-13, \cdots, 7, 3\}.$$
$$f_r^v(V) = \{2, 4, 8, 12, \cdots, 4r-4, 4r-1, 4r-2, 4r-7, 4r-11, \cdots, 9, 5\}.$$

Hence we have proved that all even cycles C_n, can be s-type, ($1 \leq s \leq r$, $n \geq 6$ and $n = 2r$), SVM labeled. □

Theorem 2.4.3. *The ladder $L_n = P_n \times P_2$, where $n \geq 3$ is SVM.*

Proof. Let $V(P_n \times P_2) = \{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ and $E(P_n \times P_2) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$. We note that the order of L_n is $2n$ and the size is $3n-2$.

The edges of L_n are labeled as follows:

$$f(u_i u_{i+1}) = \begin{cases} 3 & \text{if } i = 1 \\ 5i-1 & \text{if } i \text{ is even or } i = n-1 \\ 5i & \text{if } i \text{ is odd and } i \neq 1 \text{ and } i \neq n-1. \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} 5n-2 & \text{if } i = n-1 \\ 5i+2 & \text{if } i \neq n-1. \end{cases}$$

$$f(u_iv_i) = \begin{cases} 1 & \text{if } i = 1 \\ 5 & \text{if } i = 2 \\ 5i - 6 & \text{if } i \text{ is even and } i \neq 2 \text{ or } n \\ 5i - 5 & \text{if } i \text{ is odd and } i \neq 1 \text{ or } n \\ 5n - 4 & \text{if } i = n. \end{cases}$$

It can easily be observed that f is injective. The induced vertex labels are as follows;

$$f^v(u_i) = \begin{cases} 2 & \text{if } i = 1 \\ 5i - 4 & \text{if } 2 \leq i \leq n - 1 \\ 5n - 5 & \text{if } i = n. \end{cases}$$

$$f^v(v_i) = \begin{cases} 4 & \text{if } i = 1 \\ 5i - 2 & \text{if } 2 \leq i \leq n - 1 \\ 5n - 3 & \text{if } i = n. \end{cases}$$

It is clear that $f(E) \cup f^v(V) = \{1, 2, \cdots, 5n - 2\}$.

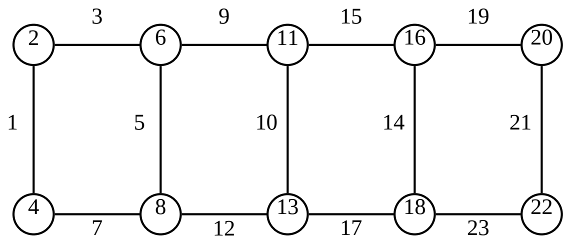
Since,

$$\begin{aligned} f(E) = \{&3, 9, 19, \cdots, 5n - 6(\text{if } n \text{ is odd}), 5n - 11(\text{if } n \text{ is even})\} \cup \\ &\{15, 25, \cdots, 5n - 10(\text{if } n \text{ is odd}), 5n - 15(\text{if } n \text{ is even})\} \cup \\ &\{7, 12, 17, \cdots, 5n - 8, 5n - 7\} \cup \\ &\{1, 5, 10, 20, \cdots, 5n - 15(\text{if } n \text{ is odd}), 5n - 10(\text{if } n \text{ is even})\} \cup \\ &\{14, 24, \cdots, 5n - 11(\text{if } n \text{ is odd}), 5n - 16(\text{if } n \text{ is even}), 5n - 4\}. \end{aligned}$$

and, $f^v(V) = \{2, 6, 11, \cdots, 5n - 9, 5n - 5, 4, 8, 13, \cdots, 5n - 7, 5n - 3\}$

Thus it is easy to verify, in both cases, that the above mentioned labeling is SVM - labeling. Hence the theorem. □

Example 2.4.4. *SVM labeling of L_5 and L_8 are shown in Figure 2.3.*



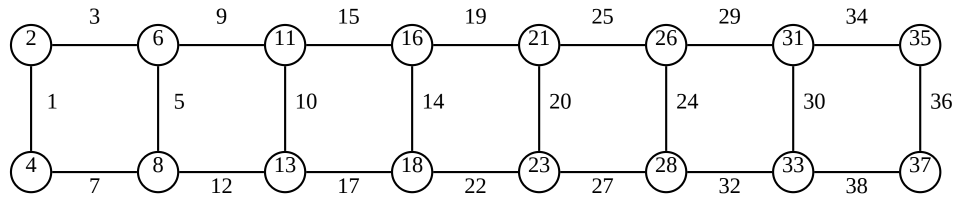


Figure 2.3: SVM labeling of L_5 and L_8 are shown.

2.5 Fans ($F_{n,}\ n \geq 2$)

Definition 2.5.1. *The fan $F_{n,}$ $(n \geq 2)$ is obtained by joining all vertices of a path P_n to a further vertex called center, and contains $n+1$ vertices and $2n-1$ edges.*

The edges of the path in a fan are named $e_{i,}$ $1 \leq i \leq n-1$, whereas the vertices of the path in a fan are named $v_{i,},\ 1 \leq i \leq n$. The center vertex is named c and the edges connecting center and the vertices of the path are named $s_{i,}$ $1 \leq i \leq n$.

2.6 Fans ($F_{n,}\ n \geq 2$) and their SVM - Behaviour

We discuss the SVM - behaviour of fans in the following three theorems.

Theorem 2.6.1. *Fans ($F_{n,}$ $n \geq 2$) are SVM - graphs, when $n \equiv 1(mod\ 2)$.*

Proof. Let ($F_{n,}$ $n \geq 2$) be a fan, where $n \equiv 1(mod\ 2)$. Let $n = 2r-1,\ r \geq 2$. We give below the SVM - labeling of F_3 and F_5 in Figure 2.4.

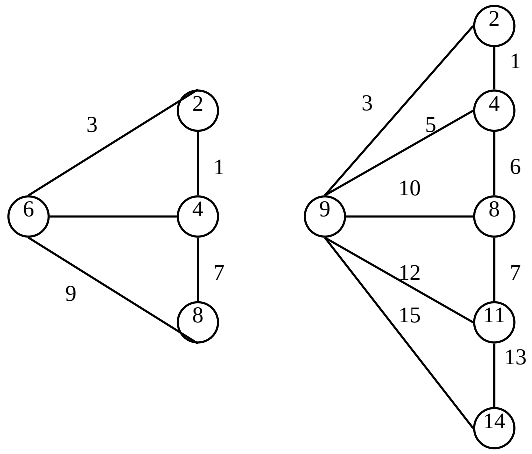


Figure 2.4: Super vertex-mean labelings of F_3 and F_5.

When $n \geq 7$, define $f : E(F_n) \rightarrow \{1, 2, 3, ..., 3n\}$ as follows:

$$f(e_i) = \begin{cases} 1, & \text{if } i = 1, \\ 3i - 1, & \text{if } 2 \leq i \leq r - 2, \\ 3i, & \text{if } i = r - 1, \\ 3i - 2, & \text{if } i = r, \\ 3i, & \text{if } r + 1 \leq i \leq n - 2, \\ 3i + 1, & \text{if } i = n - 1. \end{cases}$$

$$f(s_i) = \begin{cases} 3i, & \text{if } 1 \leq i \leq r - 2, \\ 3i - 1, & \text{if } i = r - 1, \\ 3i + 1, & \text{if } r \leq i \leq n - 2, \\ 3i, & \text{if } n - 1 \leq i \leq n. \end{cases}$$

The induced vertex labels are found to be as follows:

$$f^v(v_i) = \begin{cases} 2, & \text{if } i = 1, \\ 3i - 2, & \text{if } 2 \leq i \leq r - 1, \\ 3i - 1, & \text{if } r \leq i \leq n. \end{cases}$$

$$f^v(c) = 3r$$

Remark: While computing the value of $f^v(c)$, the exact value of it, without rounding off is found to be $3r + (\frac{n-5}{2n})$ for each $n \geq 7$. While rounding off, the value of $f^v(c)$ remains $3r$, because $\frac{n-5}{2n}$ never attains 0.5 as $\frac{n-5}{2n}$ is a converging function and converges to $\frac{1}{2}$.

Further it can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 3n\}$. Therefore $F_{n,}$ $n \geq 2$ and $n \equiv 1(mod\, 2)$ is SVM. □

Example 2.6.2. *Super vertex-mean labeling of* F_{13} *is shown in Figure* 2.5.

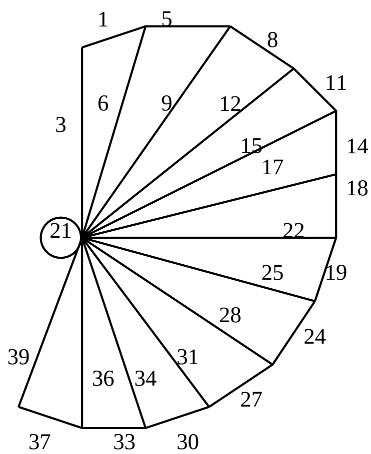


Figure 2.5: F_{13} is an SVM.

Theorem 2.6.3. *Fans* ($F_{n,}$ $n \geq 2$) *are SVM graphs, when* $n \equiv 2(mod\, 4)$.

Proof. Let ($F_{n,}$ $n \geq 2$) be a fan, where $n \equiv 2(mod\, 4)$. The SVM labeling of F_2, F_6 and F_{10} are given in Figure 2.6.

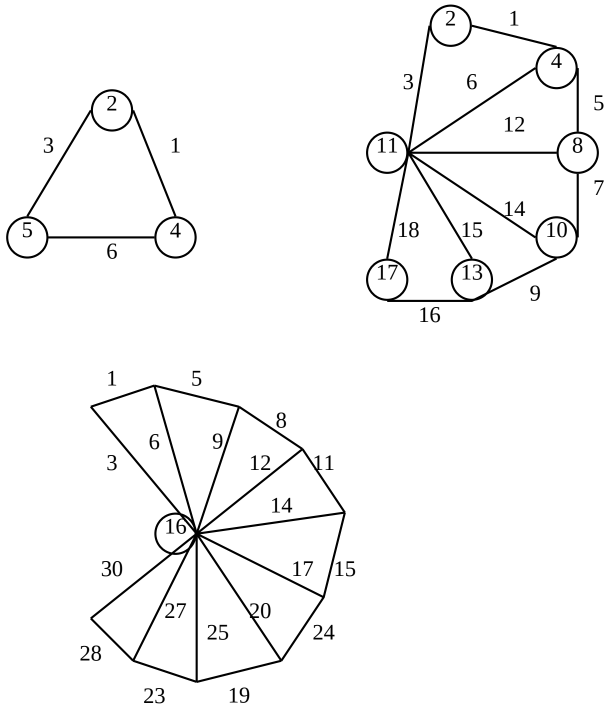


Figure 2.6: Super vertex-mean labelings of F_2, F_6 and F_{10}.

When $n \geq 14$, define $f : E(F_n) \rightarrow \{1, 2, 3, ..., 3n\}$ as follows:

$$f(e_i) = \begin{cases} 1, & \text{if } i = 1, \\ 3i - 1, & \text{if } 2 \leq i \leq \frac{n}{2} - 1, \\ 3i, & \text{if } i = \frac{n}{2}, \\ 3i + 6, & \text{if } i = \frac{n}{2} + 1, \\ 3i - 2, & \text{if } i = \frac{n}{2} + 2, \\ 3i - 1, & \text{if } i = \frac{n}{2} + 3, \\ 3i, & \text{if } \frac{n}{2} + 4 \leq i \leq n - 2, \\ 3i, & \text{if } i = n - 1 \text{ and } n = 14, \\ 3i + 1, & \text{if } i = n - 1 \text{ and } n \geq 18. \end{cases}$$

$$f(s_i) = \begin{cases} 3i, & \text{if } 1 \le i \le \frac{n}{2} - 1, \\ 3i - 1, & \text{if } i = \frac{n}{2}, \\ 3i - 2, & \text{if } i = \frac{n}{2} + 1, \\ 3i - 1, & \text{if } i = \frac{n}{2} + 2, \\ 3i + 1, & \text{if } \frac{n}{2} + 3 \le i \le n - 2, \\ 3i + 1, & \text{if } i = n - 1 \text{ and } n = 14, \\ 3i, & \text{if } i = n - 1 \text{ and } n \ge 18 \\ 3i, & \text{if } i = n. \end{cases}$$

The induced vertex labels are found to be as follows:

$$f^v(v_i) = \begin{cases} 2, & \text{if } i = 1, \\ 3i - 2, & \text{if } 2 \le i \le \frac{n}{2}, \\ 3i, & \text{if } \frac{n}{2} + 1 \le i \le \frac{n}{2} + 2, \\ 3i - 2, & \text{if } i = \frac{n}{2} + 3, \\ 3i - 1, & \text{if } \frac{n}{2} + 4 \le i \le n. \end{cases}$$

$$f^v(c) = \frac{3n + 4}{2}$$

Remark: The real value of $f^v(c)$, without rounding off is $\frac{3n+4}{2} - 0.5 + (\frac{n-16}{2n})$ for each $n \ge 18$. While rounding off, the value of $f^v(c)$ remains $\frac{3n+4}{2}$, because $\frac{n-16}{2n}$ never attains 0.5 as $\frac{n-16}{2n}$ is a converging function and converges to $\frac{1}{2}$.

Also it can be easily verified that f is a Super Vertex Mean labeling as it is an injective function and $f(E(F_n)) \cup f^v(V(F_n))$ is $\{1, 2, 3, ..., 3n\}$. Therefore $F_{n,}$ $n \ge 2$ and $n \equiv 2(mod\ 4)$ is SVM. □

Example 2.6.4. *Figure* 2.7 *gives Super vertex-mean labeling of* F_{18}.

Theorem 2.6.5. *Fans (*$F_{n,}$ $n \ge 2$*) are SVM graphs, when* $n \equiv 0(mod\ 4)$.

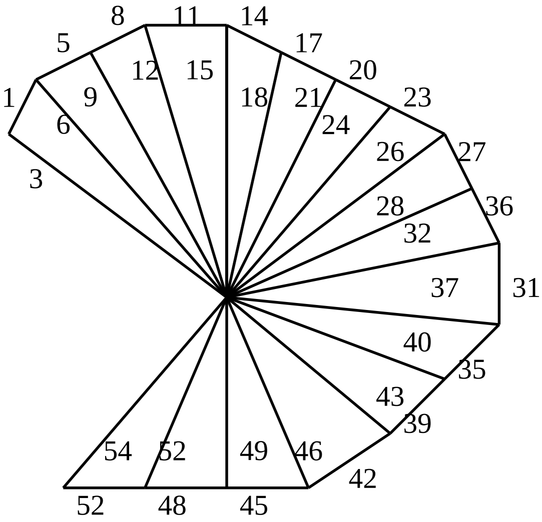


Figure 2.7: F_{18} is an SVM.

Proof. Let ($F_{n,}$ $n \geq 2$) be a fan, where $n \equiv 0 (mod\, 4)$. The SVM labeling of F_4 is illustrated in Figure 2.8.

When $n \geq 8$, define $f : E(F_n) \to \{1, 2, 3, ..., 3n\}$ as follows:

$$f(e_i) = \begin{cases} 1, & \text{if } i = 1, \\ 3i - 1, & \text{if } 2 \leq i \leq \frac{n}{2}, \\ 3i + 4, & \text{if } i = \frac{n}{2} + 1, \\ 3i, & \text{if } \frac{n}{2} + 2 \leq i \leq n - 2, \\ 3i + 1, & \text{if } i = n - 1. \end{cases}$$

$$f(s_i) = \begin{cases} 3i, & \text{if } 1 \leq i \leq \frac{n}{2}, \\ 3i + 1, & \text{if } i = \frac{n}{2} + 1, \\ 3i - 4, & \text{if } i = \frac{n}{2} + 2, \text{ and } n = 8 \text{ or } 12, \\ 3i - 5, & \text{if } i = \frac{n}{2} + 2, \text{ and } n \geq 16, \\ 3i + 1, & \text{if } \frac{n}{2} + 3 \leq i \leq n - 2, \\ 3i, & \text{if } n - 1 \leq i \leq n. \end{cases}$$

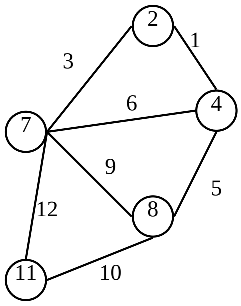


Figure 2.8: Super vertex-mean labeling of F_4.

The induced vertex labels are found to be as follows:

$$f^v(v_i) = \begin{cases} 2, & \text{if } i = 1, \\ 3i-2, & \text{if } 2 \leq i \leq \frac{n}{2}, \\ 3i, & \text{if } i = \frac{n}{2}+1, \\ 3i-1, & \text{if } \frac{n}{2}+2 \leq i \leq n. \end{cases}$$

$$f^v(c) = \begin{cases} \frac{3n+2}{2}, & \text{if } n = 8 \text{ or } 12, \\ \frac{3n+4}{2}, & \text{if } n \geq 16. \end{cases}$$

Remark: As in the previous case 2, the real value of $f^v(c)$, without rounding off is $\frac{3n+4}{2} - 0.5 + (\frac{n-16}{2n})$ for each $n \geq 16$. While rounding off, the value of $f^v(c)$ remains $\frac{3n+4}{2}$, because $\frac{n-16}{2n}$ never attains 0.5 as $\frac{n-16}{2n}$ is a converging function and converges to $\frac{1}{2}$.

It is an easy exercise to verify that f is a Super Vertex Mean labeling. f is an injective function and the union of edge labels and induced vertex labels is $\{1, 2, 3, ..., 3n\}$. Therefore $F_{n,}$ $n \geq 2$ and $n \equiv 0(mod\, 4)$ is SVM. □

Example 2.6.6. *Super vertex-mean labeling of F_{12} and F_{16} is shown in Figure 2.9.*

2.7 Super Vertex Mean Number

The concept of Super Vertex Mean Number arises from the earlier concepts such as, Mean Number, Super Mean Number etc. M.Somasundaram and R.Ponraj have introduced

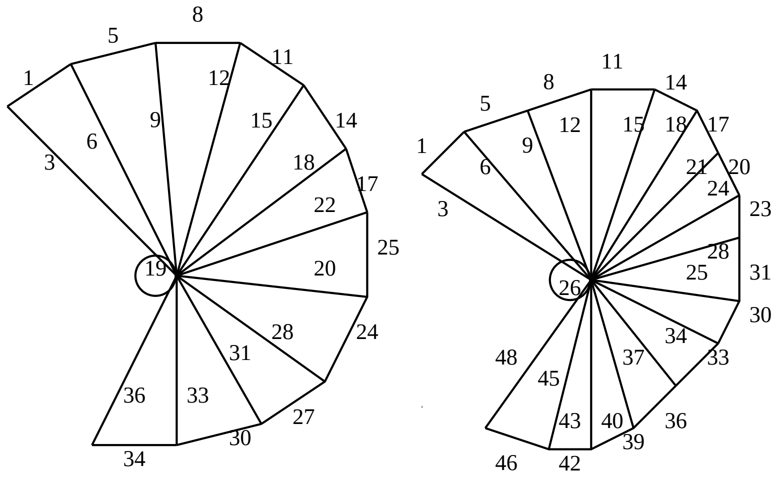


Figure 2.9: Super vertex-mean labeling of F_{12} and F_{16}.

the term Mean Number of a graph [33] and they have found the mean number of many standard graphs. Later on, A.Nagarajan et.al. introduced the concept Super Mean Number of a graph [24] and proved the existence of it for any graph by finding out the limit values of it. Encouraged by their works we introduce this new concept which we like to name as Super Vertex Mean Number or SVM - Number.

Definition 2.7.1. *Let f be a an injective function of a (p, q) - graph $G(V, E)$ defined from E to the set $\{1, 2, 3, \cdots, n\}$ that induces for each vertex v the label defined by the rule $f^v(v)$ = Round $\left(\frac{\sum_{e \in E_v} f(e)}{d(v)}\right)$, where E_v denotes the set of edges in G that are incident at the vertex v. Let $f(E) \cup f^v(V) \subseteq \{1, 2, 3, \cdots, n\}$. If n is the smallest positive integer satisfying these conditions together with the condition that all the vertex labels as well as the edge labels are distinct, then n is called the Super Vertex Mean Number (or SVM - number) of the graph $G(V, E)$, and is denoted by $SV_m(G)$.*

2.7.1 Observation

It is observed that $SV_m(G) = p + q$, for all SVM graphs G whose order is p and size is q.

And for other graphs (p, q) - graph G, $SV_m(G) \geq p + q + 1$. Therefore the lower limit of $SV_m(G)$, for any graph G is $p + q$.

For graphs containing an isolated vertex or a leaf, the Super Vertex Mean Number does not exist. i.e., for such graphs G, $SV_m(G) = \infty$. Therefore, for any (p, q) – graph G,

$$p + q \leq SV_m(G) \leq \infty$$

.

Example 2.7.2. *In Figure* $2.10.$, *it is shown that the SVM - number of* C_4, $SV_m(C_4) = 9$.

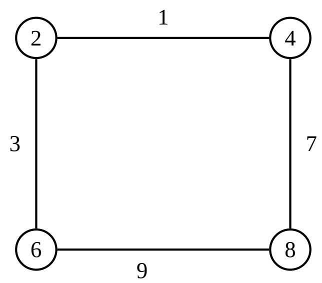


Figure 2.10: $SV_m(C_4)$ is 9

2.8 Conclusion

While analyzing the Super Vertex Mean labeling of Cycles, C_n, we observe that the ideal situation would have been that the sum of all the edge labels to be equal to the sum of all vertex labels, as the induced vertex labels are the averages of the two edge labels of the edges that are incident on the vertex and each edge is considered twice to obtain the induced vertex labels, since each cycle is a 2–regular graph.

But we notice that in the case of odd cycles, C_n, $n \equiv 1(mod\ 2)$, be it any type of SVM labeling, there are exactly two vertices which have such edges incident on it, that are labeled with two integers one of which is odd and the other is even. Therefore the induced vertex labels of these two vertices are 0.5 each more than the actual average of the labels of the incident edges on it, as per the definition of the SVM labeling (due to the rounding off factor). When we sum up all the induced vertex labels, we get an integer which is exactly

one more than the sum of all the edge labels. Or in other words, this sum of all induced vertex labels is 0.5 more than the half of the sum of first $2n$ positive integers. Similarly the sum of all edge labels is 0.5 less than the half of the sum of the first $2n$ positive integers.
We also know that the half of the sum the of first $2n$ positive integers is

$$\frac{(2n)(2n+1)}{4}$$

For example, type 2 labeling of C_5, where, $2n = 10$, and
Half of the sum of first 10 positive integers =

$$\frac{10 \times 11}{4} = 27.5$$

The sum of the vertex labels is $2 + 4 + 8 + 9 + 5 = 28$, and
The sum of the edge labels is $1 + 6 + 10 + 7 + 3 = 27$.
Therefore, the sum of the vertex labels for $C_n, n \equiv 1(mod\ 2)$, is given by the following equation,

$$\sum_{i=1}^{n} f^v(v_i) = (\frac{(2n)(2n+1)}{4} + 0.5)$$

and,
the sum of the edge labels for $C_n, n \equiv 1(mod\ 2)$, is

$$\sum_{i=1}^{n} f(e_i) = (\frac{(2n)(2n+1)}{4} - 0.5).$$

On the same note, for even cycles, $C_n, n \equiv 0(mod\ 2)$, there are exactly 4 vertices which have edges incident on them in such a manner that they are labeled with integers of which one is odd and the other is even, resulting in an increase of 2 in the sum of the vertex labels to that of the edge labels.
Therefore, sum of the edge labels = sum of the vertex labels -2
Also, sum of the first $2n$ positive integers = $\frac{(2n)(2n+1)}{2}$
So, sum of the vertex labels $= \frac{(2n)(2n+1)}{2}$ - sum of the edge labels
i.e., $= \frac{(2n)(2n+1)}{2}$ - sum of the vertex labels $+2$
i.e., $2\times$ sum of the vertex labels $= \frac{(2n)(2n+1)}{2} + 2$
Therefore the sum of the vertex labels of $C_n, n \equiv 0(mod\ 2)$, is given by the following

equation,

$$\sum_{i=1}^{n} f^v(v_i) = (\frac{(2n)(2n+1)}{4} + 1).$$

and sum of the edge labels for $C_n, n \equiv 0(mod\, 2)$ is

$$\sum_{i=1}^{n} f(e_i) = (\frac{(2n)(2n+1)}{4} - 1).$$

We conclude by stating that the above equations are not sufficient but necessary conditions for a set of integers from the set of first $2n$ positive integers to be the edge label set, $f(E)$ or the induced vertex label set, $f^v(V)$ of a Super Vertex Mean labeling of any type for any Cycle C_n. It is given as follows,

$$\sum_{i=1}^{n} f(e_i) = \begin{cases} (\frac{(2n)(2n+1)}{4} - 0.5) & \text{if } n \equiv 1(mod\, 2) \\ (\frac{(2n)(2n+1)}{4} - 1) & \text{if } n \equiv 0(mod\, 2). \end{cases}$$

$$\sum_{i=1}^{n} f^v(v_i) = \begin{cases} (\frac{(2n)(2n+1)}{4} + 0.5) & \text{if } n \equiv 1(mod\, 2) \\ (\frac{(2n)(2n+1)}{4} + 1) & \text{if } n \equiv 0(mod\, 2). \end{cases}$$

Also all the fans ($F_{n,}\ n \geq 2$) are SVM graphs. They are categorized into three cases, of which the first includes fans whose n is odd and the last two cases together form those whose n is even. The reader is further encouraged to explore the possibilities of proving that all wheels $(W_{n,}\ n \geq 3)$ are SVM graphs. The wheel $W_{n,}\ n \geq 3$ is obtained by joining all vertices of a cycle C_n to a further vertex called center, and contains $n+1$ vertices and $2n$ edges. Wheels have a lot in common with fans when we study their SVM - behaviour.

Chapter 3

Cyclic Snakes as SVM - graphs

A chain of cycles is known as a cyclic snake. Cyclic snakes can be constructed in a variety of ways. This chapter presents one type of cyclic snakes and proves that all of them are SVM -graphs. Before entering into the results, we define the term Cyclic Snakes and introduce the type that we examine in this chapter.

3.1 Cyclic Snakes

Definition 3.1.1. *A kC_n - snake has been defined as a connected graph in which all the blocks are isomorphic to the cycle C_n and the block-cut point graph is a path P, where P is the path of minimum length that contains all the cut vertices of a kC_n - snake. Barrientos [4] has proved that any kC_n – snake is represented by a string $s_1, s_2, s_3, \cdots, s_{k-2}$ of integers of length $k-2$, where the i^{th} integer, s_i on the string is the distance between i^{th} and $i+1^{th}$ cut vertices along the path, P, from one extreme and is taken from $S_n = \{1, 2, 3, \cdots, \lfloor \frac{n}{2} \rfloor\}$.*

Remark: The strings obtained for both the extremes are assumed to be the same. In this chapter we consider only those Cyclic snakes with $s_i = 1$, for all $1 \leq i \leq k-2$.

3.1.1 Known Results

- **Result 1.** All the cycles except C_4 are SVM - graphs [Theorem 2.2.1]

- **Result 2.** All odd cycles can be SVM - labeled as many as $\lfloor \frac{n}{2} \rfloor$ different ways and every even cycle, except C_4, can have $\left(\lfloor \frac{n}{2} \rfloor - 1 \right)$ types of SVM - labeling [Theorem 2.4.1 & Theorem 2.4.2]

3.1.2 Triangular Snakes

Theorem 3.1.2. *A triangular snake with k blocks is an SVM - graph*

Proof. Let kC_3 be a triangular snake with k blocks with p vertices and q edges. Then $p = 2k + 1$ and $q = 3k$. Let $V(kC_3) = \{u_i : 1 \leq i \leq k+1\} \cup \{v_i : 1 \leq i \leq k\}$ and $E(kC_3) = \{u_i u_{i+1}, u_i v_i, v_i u_{i+1} : 1 \leq i \leq k\}$.
The edges of kC_3 are labeled as follows:

$$f(u_i u_{i+1}) = \begin{cases} 1 & \text{if } i = 1 \\ 5i & \text{if } i \text{ is even and } i \neq k \\ 5i - 3 & \text{if } i \text{ is odd and } i \neq 1 \\ 5k + 1 & \text{if } k \text{ is even and } i = k \end{cases}$$

$$f(u_i v_i) = \begin{cases} 5i - 3 & \text{if } i \text{ is even} \\ 5i - 2 & \text{if } i \text{ is odd} \end{cases}$$

$$f(v_i u_{i+1}) = \begin{cases} 5i & \text{if } i \text{ is odd and } i \neq k \\ 5i - 1 & \text{if } i \text{ is even} \\ 5k + 1 & \text{if } k \text{ is odd and } i = k \end{cases}$$

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2 & \text{if } i = 1 \\ 5i - 4 & \text{if } 2 \leq i \leq k \\ 5i - 6 & \text{if } i = k + 1 \text{ and k is odd} \\ 5i - 5 & \text{if } i = k + 1 \text{ and k is even} \end{cases}$$

$$f^v(v_i) = \begin{cases} 5i-2 & \text{if } i \text{ is even} \\ 5i-1 & \text{if } i \text{ is odd and } i \neq k \\ 5k & \text{if } i = k \text{ is odd} \end{cases}$$

It can be easily verified that f is injective and the set of edge labels and induced vertex labels is $\{1, 2, ..., 5k+1\}$. □

Example 3.1.3. *Super vertex mean labeling of triangular snakes is shown in Figure* 3.1.

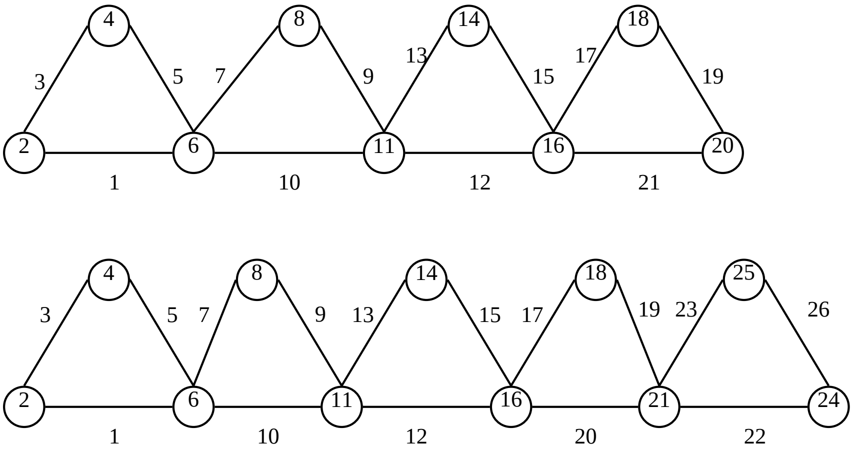


Figure 3.1: Super vertex mean labeling of Triangular snakes

3.1.3 Quadrilateral Snakes

Theorem 3.1.4. *Quadrilateral snakes with $k \geq 2$ blocks and each $s_i = 1$ are SVM - graphs*

Proof. Let kC_4 be a quadrilateral snake with $V(kC_4) = \{u_i : 1 \leq i \leq k+1\} \cup \{u_i, w_i : 1 \leq i \leq k\}$ and $E(kC_4) = \{u_iu_{i+1}, u_iv_i, u_{i+1}w_i, v_iw_i : 1 \leq i \leq k\}$. Then $p = 3k+1$ and $q = 4k$.

Define $f : E(kC_4) \to \{1, 2, 3, ..., 7k+1\}$ as follows:

$$f(u_iu_{i+1}) = \begin{cases} 7i & \text{if } 1 \leq i \leq k-1 \\ 7k+1 & \text{if } i = k. \end{cases}$$

$$f(u_iv_i) = 7i - 6 \text{ if } i \ \leq i \leq k.$$

$$f(v_iw_i) = \begin{cases} 3 & \text{if } i = k \\ 7i-3 & \text{if } 2 \leq i \leq k. \end{cases}$$

$$f(w_iu_{i+1}) = 7i - 1 \text{ if } i \ \leq i \leq k.$$

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 4 & \text{if } i = 1 \\ 7k & \text{if } i = k+1 \\ 7i-5 & \text{otherwise.} \end{cases}$$

$$f^v(v_i) = \begin{cases} 2 & \text{if } i = 1 \\ 7i-4 & \text{otherwise.} \end{cases}$$

$$f^v(w_i) = 7i - 2 \text{ if } 1 \leq i \leq k.$$

It can be easily verified that f is injective and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 7k + 1\}$. □

Example 3.1.5. *A Super vertex-mean labeling of a Quadrilateral snake ($4C_4$) is shown in Figure 3.2.*

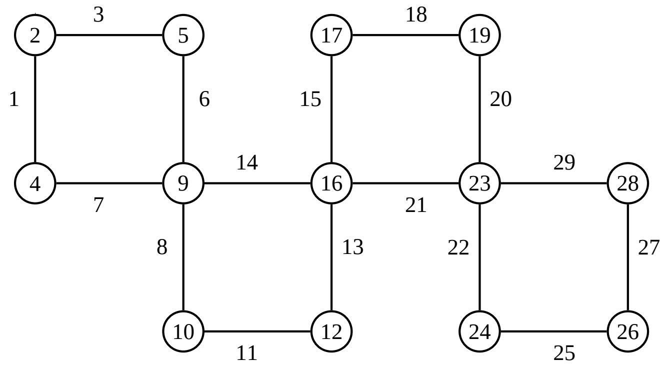


Figure 3.2: A Super vertex-mean labeling of a Quadrilateral snake, $4C_4$

3.2 Cyclic Snakes of cycles of higher orders

We proceed to prove that cyclic snakes of cycles of the other orders are also SVM - graphs.

3.2.1 Pentagonal Snakes

Theorem 3.2.1. *Pentagonal snakes with k blocks and each $s_i = 1$ are SVM -graphs.*

Proof. Let kC_5 be a pentagonal snake with k blocks of C_5.
Let $V(kC_5) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq 5\}$ and
$E(kC_5) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ and $e_{i,5} = v_{i,5}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq 4\}$.
Note that $v_{i,5} = v_{i+1,1}$ for $1 \leq i \leq k-1$, and we refer this vertex as $v_{i,5}$ throughout this proof.
Now, $p = 4k + 1$, $q = 5k$ and $p + q = 9k + 1$.
Define $f : E(kC_5) \to \{1, 2, 3, \cdots, 9k + 1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 2j-1, & \text{if } i=1, \text{ and } 1 \leq j \leq 3 \\ 2j, & \text{if } i=1, \text{ and } 4 \leq j \leq 5 \\ 9i-9, & \text{if } 2 \leq i \leq k \text{ and } j = 1 \\ 9i+2j-9, & \text{if } 2 \leq i \leq k \text{ and } 2 \leq j \leq 5. \end{cases}$$

It can be easily verified that f is injective.
Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} n+1, & \text{if } i=1, \text{ and } j = 1 \\ 2j-2, & \text{if } i=1, \text{ and } 2 \leq j \leq 3 \\ 2j-1, & \text{if } i=1, \text{ and } j = 4 \\ 9i+3, & \text{if } 1 \leq i \leq k-1 \text{ and } j = 5 \end{cases}$$

$$f^v(v_{i,j}) = \begin{cases} 9i-7, & \text{if } 2 \le i \le k \text{ and } j = 2 \\ 9i+2j-10, & \text{if } 2 \le i \le k, \text{ and } 3 \le j \le 5 \\ 9k, & \text{if } i = k, \text{ and } j = 5. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 9k+1\}$.

Therefore, pentagonal snakes kC_5 with each $s_i = 1$ are Super Vertex Mean graphs. □

Example 3.2.2. *In Figure 3.3. we have an SVM labeling of a pentagonal snake with* 4 *blocks.*

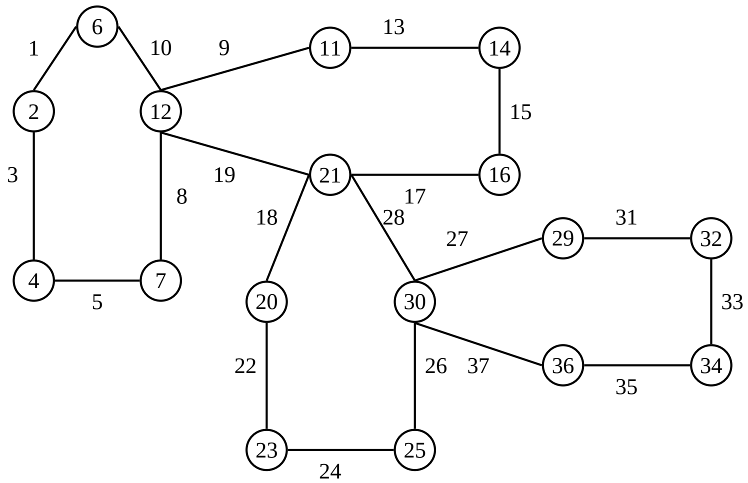


Figure 3.3: Super vertex-mean labeling of a Pentagonal snake with 4 blocks

3.2.2 Hexagonal Snakes

Theorem 3.2.3. *Hexagonal snakes with each* $s_i = 1, 1 \le i \le k-2$ *are Super Vertex Mean Graphs.*

Proof. Let kC_6 be a hexagonal snake with k blocks of C_6.
Let $V(kC_6) = \{v_{i,j}; 1 \le i \le k, 1 \le j \le 6\}$ and
$E(kC_6) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ and $e_{i,6} = v_{i,6}v_{i,1}; 1 \le i \le k, 1 \le j \le 5\}$.
Note that $v_{i,6} = v_{i+1,1}$ for $1 \le i \le k-1$, and we refer this vertex as $v_{i,6}$ throughout this

proof.

Now, $p = 5k + 1$ and $q = 6k$ and $p + q = 11k + 1$.

Define $f : E(G_n) \to \{1, 2, 3, \cdots, 11k + 1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 9 - 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 2 \\ 6j - 17, & \text{if } i = 1, \text{ and } 3 \leq j \leq 4 \\ 27 - 3j, & \text{if } i = 1, \text{ and } 5 \leq j \leq 6 \\ 11i - 11, & \text{if } 2 \leq i \leq k, \text{ and } j = 1 \\ 11i + 2j - 11, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq 6. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 11 - 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3 \\ 6j - 20, & \text{if } i = 1, \text{ and } 4 \leq j \leq 5 \\ 11i + 3, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 6 \\ 11i - 9, & \text{if } 2 \leq i \leq k, \text{ and } j = 2 \\ 11i + 2j - 12, & \text{if } 2 \leq i \leq k, \text{ and } 3 \leq j \leq 5 \\ 11k, & \text{if } i = k, \text{ and } j = 6. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 11k + 1\}$.

Therefore, hexagonal snakes with k blocks of C_6 are Super Vertex Mean graphs. □

Example 3.2.4. *SVM labeling of a hexagonal snake with* 3 *blocks is given in Figure* 3.4.

3.2.3 kC_n Snakes, $n \geq 7$ and $n \equiv 3(mod\ 4)$

Theorem 3.2.5. *Let* kC_n *be a cyclic snake with* k *blocks of* $C_n, n \geq 7$ *and* $n \equiv 3(mod\ 4)$. *Then* kC_n *is a Super Vertex Mean graph.*

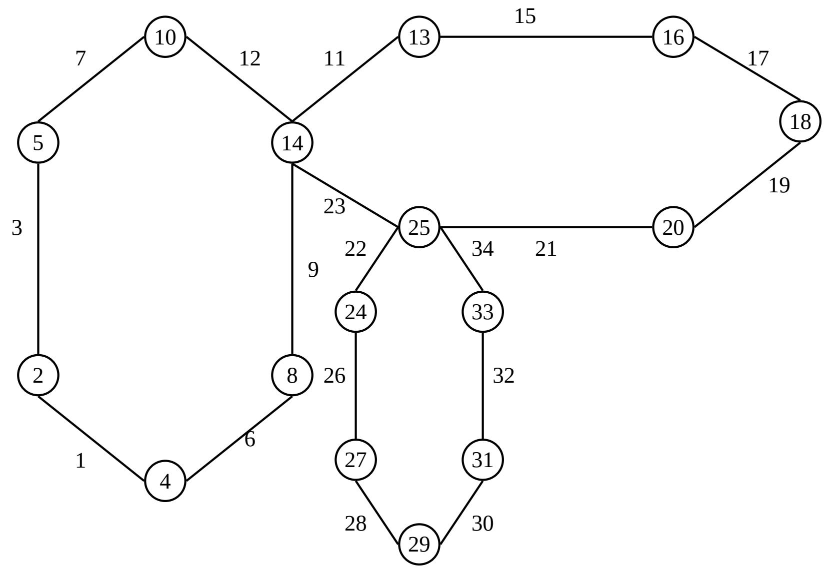


Figure 3.4: Hexagonal snake with 3 blocks is SVM

Proof. Let kC_n be a cyclic snake with k blocks of $C_n, n \geq 7$ and $n \equiv 3(mod\ 4)$.
Let $n = 2r + 1$, and $r = 2s + 1$ so that $n = 4s + 3$.
Let $V(kC_n) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq n\}$ and
$E(kC_n) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ & $e_{i,n} = v_{i,n}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq n-1\}$.
Note that $v_{i,n} = v_{i+1,1}$ for $1 \leq i \leq k-1$, and we refer this vertex as $v_{i,n}$ throughout this proof.
Now, $p = (n-1)k + 1$ and $q = nk$ and $p + q = (2n-1)k + 1$.
Define $f : E(kC_n) \to \{1, 2, 3, \cdots, (2n-1)k + 1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 2j-1, & \text{if } i = 1, \text{ and } 1 \leq j \leq r+1 \\ 2j, & \text{if } i = 1, \text{ and } r+2 \leq j \leq n \\ (2n-1)i - (2n-1), & \text{if } 2 \leq i \leq k \text{ and } j = 1 \\ (2n-1)i + 2j - (2n), & \text{if } 2 \leq i \leq k \text{ and } 2 \leq j \leq r-s \\ (2n-1)i + 2j - (2n-1), & \text{if } 2 \leq i \leq k \text{ and } r-s+1 \leq j \leq n. \end{cases}$$

$$f(e_{i,j}) = \begin{cases} 2j-1, & \text{if } i=1, \text{ and } 1 \le j \le 2s+2 \\ 2j, & \text{if } i=1, \text{ and } 2s+3 \le j \le 4s+3 \\ (8s+5)i-(8s+5), & \text{if } 2 \le i \le k \text{ and } j=1 \\ (8s+5)i+2j-(8s+6), & \text{if } 2 \le i \le k \text{ and } 2 \le j \le s+1 \\ (8s+5)i+2j-(8s+5), & \text{if } 2 \le i \le k \text{ and } s+2 \le j \le 4s+3. \end{cases}$$

And, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} n+1, & \text{if } i=1, \text{ and } j=1 \\ 2j-2, & \text{if } i=1, \text{ and } 2 \le j \le r+1 \\ 2j-1, & \text{if } i=1 \text{ and } r+2 \le j \le n-1 \\ (2n-1)i+r+1, & \text{if } 1 \le i \le k-1 \text{ and } j=n \\ (2n-1)i+2j-(2n+1), & \text{if } 2 \le i \le k \text{ and } 2 \le j \le r-s \\ (2n-1)i+2j-(2n), & \text{if } 2 \le i \le k \text{ and } r-s+1 \le j \le n-1 \\ (2n-1)k & \text{if } i=k \text{ and } j=n. \end{cases}$$

$$= \begin{cases} 4s+4, & \text{if } i=1, \text{ and } j=1 \\ 2j-2, & \text{if } i=1, \text{ and } 2 \le j \le 2s+2 \\ 2j-1, & \text{if } i=1 \text{ and } 2s+3 \le j \le 4s+2 \\ (8s+5)i+2s+2, & \text{if } 1 \le i \le k-1 \text{ and } j=4s+3 \\ (8s+5)i+2j-(8s+7), & \text{if } 2 \le i \le k \text{ and } 2 \le j \le s+1 \\ (8s+5)i+2j-(8s+6), & \text{if } 2 \le i \le k \text{ and } s+2 \le j \le 4s+2 \\ (8s+5)k & \text{if } i=k \text{ and } j=4s+3. \end{cases}$$

We prove the theorem by using mathematical induction on s.

When $s = 1, r = 3$ and $n = 7$ and the cyclic snake is a heptagonal snake with k cycles of C_7.

Now, $p = 7k + 1$ and $q = 7k$ and $p + q = 13k + 1$.

Define $f : E(kC_n) \rightarrow \{1, 2, 3, \cdots, 13k + 1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 2j - 1, & \text{if } i = 1, \text{ and } 1 \leq j \leq 4 \\ 2j, & \text{if } i = 1, \text{ and } 5 \leq j \leq 7 \\ 13i - 13, & \text{if } 2 \leq i \leq k, \text{ and } j = 1 \\ 13i + 2j - 14, & \text{if } 2 \leq i \leq k, \text{ and } j = 2 \\ 13i + 2j - 13, & \text{if } 2 \leq i \leq k, \text{ and } 3 \leq j \leq 7. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} n + 1, & \text{if } i = 1, \text{ and } j = 1 \\ 2j - 2, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4 \\ 2j - 1, & \text{if } i = 1, \text{ and } 5 \leq j \leq 6 \\ 13i + 14, & \text{if } 1 \leq i \leq k - 1, \text{ and } j = 7 \\ 13i - 11, & \text{if } 2 \leq i \leq k, \text{ and } j = 2 \\ 13i + 2j - 14, & \text{if } 2 \leq i \leq k \text{ and } 3 \leq j \leq 6 \\ 13k, & \text{if } i = k, \text{ and } j = 7. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 13k + 1\}$.

Let,

$$A_1 = \{2j - 1, i = 1 \ \& \ 1 \leq j \leq 4\},$$

$$A_2 = \{2j, i = 1 \ \& \ 5 \leq j \leq 7\},$$

$$A_3 = \{13i - 13, 2 \leq i \leq k \ \& \ j = 1\},$$

$$A_4 = \{13i + 2j - 14, 2 \leq i \leq k \ \& \ j = 2\},$$

$$A_5 = \{13i + 2j - 13, 2 \leq i \leq k \ \& \ 3 \leq j \leq 7\}.$$

And let,

$$B_1 = \{8\},$$
$$B_2 = \{2, 4, 6\},$$
$$B_3 = \{9, 11\},$$
$$B_4 = \{17, 30, 43, 56, \cdots, 13k - 22, 13k - 9\},$$
$$B_5 = \{15, 28, 41, \cdots, 13k - 24, 13k - 11\},$$
$$B_6 = \{18, 20, 22, 24, \cdots, 13k - 8, 13k - 6, 13k - 4, 13k - 2\},$$
$$B_7 = \{13k\}.$$

$$\begin{aligned} A_1 \cup B_2 \cup B_1 \cup B_3 \cup A_2 &= \{1, 2, 3, 4 \cdots, 11, 12, 14\}, \\ A_3 &= \{13, 26, 39, \cdots, 13k - 13\}, \\ B_5 \cup A_4 \cup B_4 \cup B_6 \cup A_5 &= \{15, 16, \cdots, 24, 25, 27, 28, \cdots, \\ &\quad 38, 40, \cdots, 13k - 1, 13k + 1\}, \end{aligned}$$

$$\begin{aligned} A_1 \cup B_2 \cup B_1 \cup B_3 \cup A_2 \cup A_3 \cup B_5 \cup A_4 \cup \\ B_4 \cup B_6 \cup A_5 \cup B_7 \\ = \{1, 2, 3, \cdots, 13k - 1, 13k, 13k + 1\}. \end{aligned}$$

Thus the theorem is true when $s = 1$.

Now we assume that the theorem is true for $s - 1$ (i.e., for $r - 2$ and $n - 4$). The induction hypothesis is that the edge labeling,

$$f : E(kC_{n-4}) \to \{1, 2, 3, \cdots, (2n - 9)k + 1\},$$

defined as follows, is a Super Vertex Mean Labeling, where $n \geq 11$ and $n \equiv 3(mod\, 4)$ and $k \geq 2$.

$$f(e_{i,j}) = \begin{cases} 2j-1, & \text{if } i=1, \text{ and } 1 \le j \le r-1 \\ 2j, & \text{if } i=1, \text{ and } r \le j \le n-4 \\ (2n-9)i-(2n-9), & \text{if } 2 \le i \le k \text{ and } j=1 \\ (2n-9)i+2j-(2n-8), & \text{if } 2 \le i \le k \text{ and } 2 \le j \le r-s-1 \\ (2n-9)i+2j-(2n-9), & \text{if } 2 \le i \le k \text{ and } r-s \le j \le n-4. \end{cases}$$

$$= \begin{cases} 2j-1, & \text{if } i=1, \text{ and } 1 \le j \le 2s \\ 2j, & \text{if } i=1, \text{ and } 2s+1 \le j \le 4s-1 \\ (8s-3)i-(8s-3), & \text{if } 2 \le i \le k \text{ and } j=1 \\ (8s-3)i+2j-(8s-2), & \text{if } 2 \le i \le k \text{ and } 2 \le j \le s \\ (8s-3)i+2j-(8s-3), & \text{if } 2 \le i \le k \text{ and } s+1 \le j \le 4s-1. \end{cases}$$

Now we prove that the result is true for any s. If we replace s with $s+1$ in the above mappings we get,

$$f(e_{i,j}) = \begin{cases} 2j-1, & \text{if } i=1, \text{ and } 1 \le j \le 2s+2 \\ 2j, & \text{if } i=1, \text{ and } 2s+3 \le j \le 4s+3 \\ (8s+5)i-(8s+5), & \text{if } 2 \le i \le k \text{ and } j=1 \\ (8s+5)i+2j-(8s+6), & \text{if } 2 \le i \le k \text{ and } 2 \le j \le s+1 \\ (8s+5)i+2j-(8s+5), & \text{if } 2 \le i \le k \text{ and } s+2 \le j \le 4s+3. \end{cases}$$

$$= \begin{cases} 2j-1, & \text{if } i=1, \text{ and } 1 \le j \le r+1 \\ 2j, & \text{if } i=1, \text{ and } r+2 \le j \le n \\ (2n-1)i-(2n-1), & \text{if } 2 \le i \le k \text{ and } j=1 \\ (2n-1)i+2j-(2n), & \text{if } 2 \le i \le k \text{ and } 2 \le j \le r-s \\ (2n-1)i+2j-(2n-1), & \text{if } 2 \le i \le k \text{ and } r-s+1 \le j \le n. \end{cases}$$

And, the induced vertex label is,

$$f^v(v_{i,j}) = \begin{cases} 4s+4, & \text{if } i=1, \text{ and } j=1 \\ 2j-2, & \text{if } i=1, \text{ and } 2 \le j \le 2s+2 \\ 2j-1, & \text{if } i=1 \text{ and } 2s+3 \le j \le 4s+2 \\ (8s+5)i+2s+2, & \text{if } 1 \le i \le k-1 \text{ and } j=4s+3 \\ (8s+5)i+2j-(8s+7), & \text{if } 2 \le i \le k \text{ and } 2 \le j \le s+1 \\ (8s+5)i+2j-(8s+6), & \text{if } 2 \le i \le k \text{ and } s+2 \le j \le 4s+2 \\ (8s+5)k & \text{if } i=k \text{ and } j=4s+3. \end{cases}$$

$$= \begin{cases} n+1, & \text{if } i=1, \text{ and } j=1 \\ 2j-2, & \text{if } i=1, \text{ and } 2 \le j \le r+1 \\ 2j-1, & \text{if } i=1 \text{ and } r+2 \le j \le n-1 \\ (2n-1)i+r+1, & \text{if } 1 \le i \le k-1 \text{ and } j=n \\ (2n-1)i+2j-(2n+1), & \text{if } 2 \le i \le k \text{ and } 2 \le j \le r-s \\ (2n-1)i+2j-(2n), & \text{if } 2 \le i \le k \text{ and } r-s+1 \le j \le n-1 \\ (2n-1)k & \text{if } i=k \text{ and } j=n. \end{cases}$$

It is clear that $f(E) \cup f^v(V) = \{1, 2, 3, \cdots, (2n)k, (2n-1)k, (2n-1)k+1\}$

Thus the theorem is proved by Mathematical Induction. □

Example 3.2.6. *SVM Labeling of Undecagonal snake with* 4 *blocks of* C_{11} *is shown in Figure 3.5.*

3.2.4 kC_n Snakes, $n \ge 8$ and $n \equiv 0(mod\ 4)$

Theorem 3.2.7. *Let* kC_n *be a cyclic snake with* k *blocks of* $C_n, n \ge 8$ *and* $n \equiv 0(mod\ 4)$. *Then* kC_n *is a Super Vertex Mean graph.*

Proof. Let kC_n be a cyclic snake with k blocks of $C_n, n \ge 8$ and $n \equiv 0(mod\ 4)$.

Let $n = 2r$, and $r = 2s$ so that $n = 4s$.

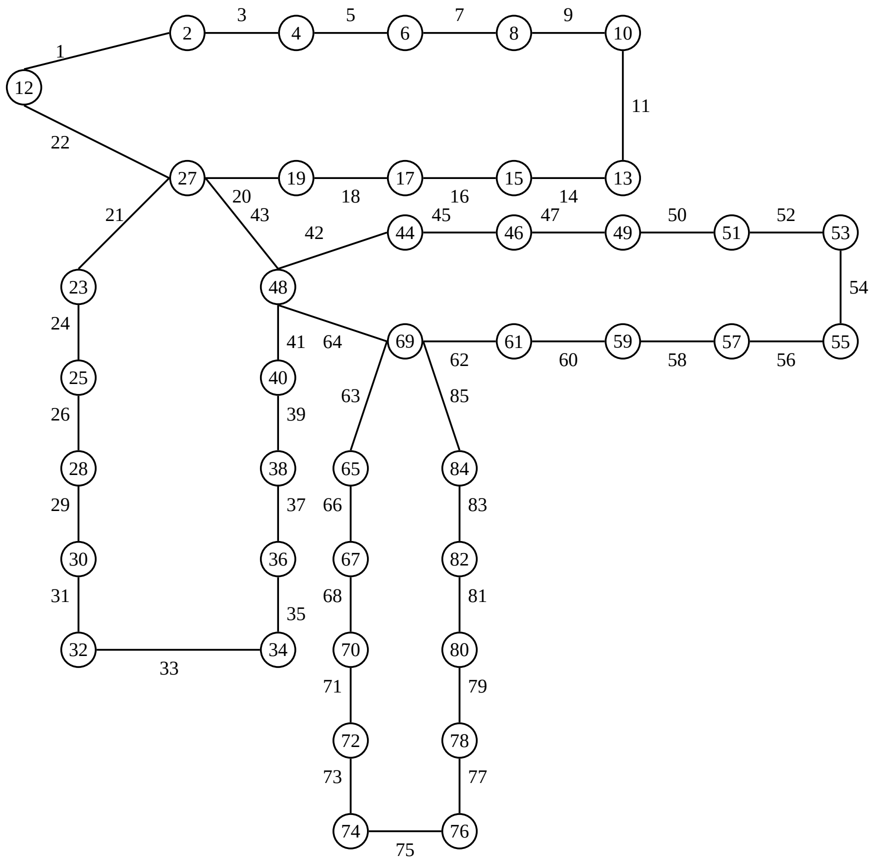


Figure 3.5: SVM labeling of an Undecagonal (C_{11}) snake with 4 blocks

Let $V(kC_n) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq n\}$ and
$E(kC_n) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ & $e_{i,n} = v_{i,n}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq n-1\}$.

Note that $v_{i,n} = v_{i+1,1}$ for $1 \leq i \leq k-1$, and we refer this vertex as $v_{i,n}$ throughout this proof.

Now, $p = (n-1)k + 1$ and $q = nk$ and $p + q = (2n-1)k + 1$.

Define $f : E(kC_n) \to \{1, 2, 3, \cdots, (2n-1)k+1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 2n-2j-2, & \text{if } i=1, \text{ and } 1 \le j \le r-4 \\ 2n-2j-3, & \text{if } i=1, \text{ and } r-3 \le j \le n-6 \\ 3n-3j-3, & \text{if } i=1, \text{ and } n-5 \le j \le n-4 \\ 1, & \text{if } i=1, \text{ and } j=n-3 \\ 7, & \text{if } i=1, \text{ and } j=n-2 \\ 4n-2j-2, & \text{if } i=1, \text{ and } n-1 \le j \le n \\ (2n-1)i-(2n-1), & \text{if } 2 \le i \le k \text{ and } j=1 \\ (2n-1)i+2j-2n, & \text{if } 2 \le i \le k \text{ and } 2 \le j \le s \\ (2n-1)i+2j-(2n-1), & \text{if } 2 \le i \le k \text{ and } s+1 \le j \le n. \end{cases}$$

And, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2n-2j-1, & \text{if } i=1, \text{ and } 1 \le j \le r-3 \\ 2n-2j-2, & \text{if } i=1, \text{ and } r-2 \le j \le n-5 \\ 5, & \text{if } i=1, \text{ and } j=n-4 \\ 8+2j-2n, & \text{if } i=1, \text{ and } n-3 \le j \le n-2 \\ n+4, & \text{if } i=1, \text{ and } j=n-1 \\ (2n-1)i+r, & \text{if } 1 \le i \le k-1 \text{ and } j=n \\ (2n-1)i+2j-(2n+1), & \text{if } 2 \le i \le k \text{ and } 2 \le j \le s \\ (2n-1)i+2j-2n, & \text{if } 2 \le i \le k \text{ and } s+1 \le j \le n-1 \\ (2n-1)k, & \text{if } i=k \text{ and } j=n. \end{cases}$$

It can be easily proved using mathematical induction on s as in the above theorem that the labeling $f : E(kC_n) \to \{1, 2, 3, \ldots, (2n-1)k+1\}$ is an SVM labeling.

Hint: Wherever r and n appear, we need to change those variables into s using $n = 4s$ and $r = 2s$. □

Example 3.2.8. *A Dodeagonal* (C_{12}) *Snake with* 4 *blocks is an SVM graph as shown in Figure* 3.6.

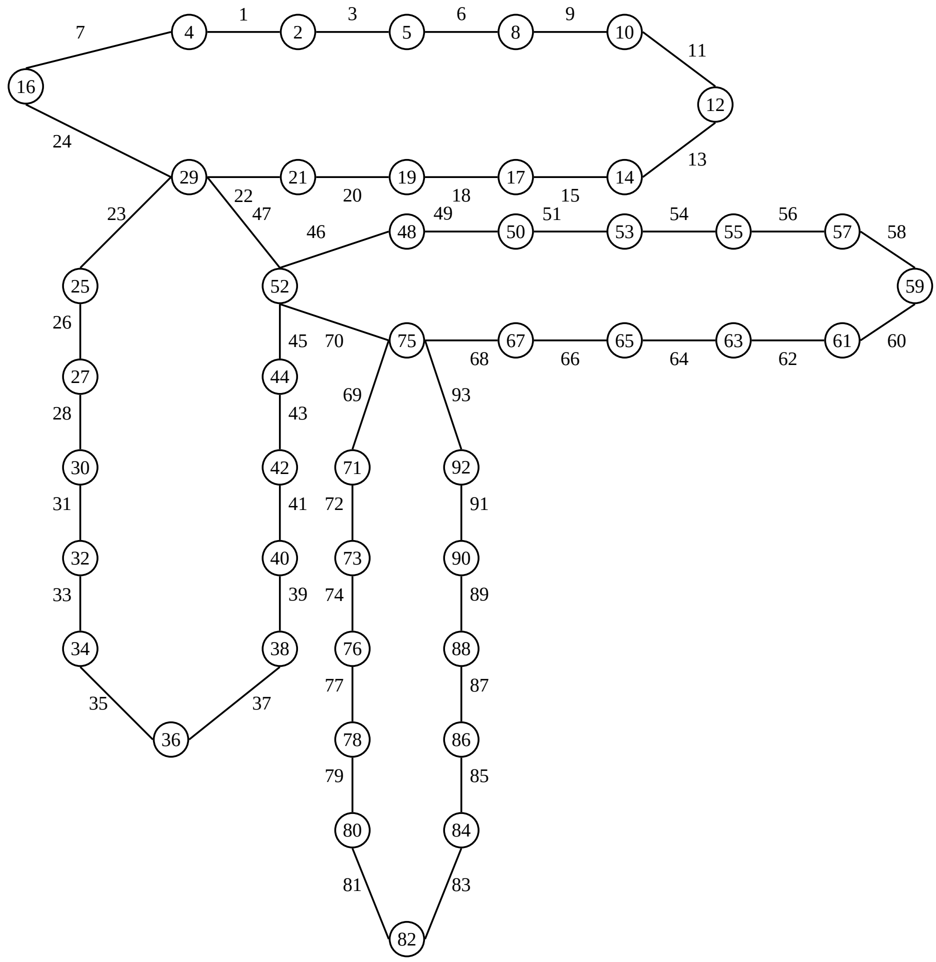


Figure 3.6: SVM labeling of a Dodecagonal (C_{12}) snake with 4 blocks.

3.2.5 kC_n Snakes, $n \geq 9$ and $n \equiv 1(mod\ 4)$

Theorem 3.2.9. *Let* kC_n *be a cyclic snake with* k *blocks of* $C_n, n \geq 9$ *and* $n \equiv 1(mod\ 4)$. *Then* kC_n *is a Super Vertex Mean graph.*

Proof. Let kC_n be a cyclic snake with k blocks of $C_n, n \geq 9$ and $n \equiv 1(mod\ 4)$.

Let $n = 2r + 1$, and $r = 2s$ so that $n = 4s + 1$.

Let $V(kC_n) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq n\}$ and
$E(kC_n) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ & $e_{i,n} = v_{i,n}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq n-1\}$.
Note that $v_{i,n} = v_{i+1,1}$ for $1 \leq i \leq k-1$, and we refer this vertex as $v_{i,n}$ throughout this proof.
Now, $p = (n-1)k + 1$ and $q = nk$ and $p + q = (2n-1)k + 1$.
Define $f : E(kC_n) \rightarrow \{1, 2, 3, \cdots, (2n-1)k + 1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 2j-1, & \text{if } i = 1, \text{ and } 1 \leq j \leq r+1 \\ 2j, & \text{if } i = 1, \text{ and } r+2 \leq j \leq n \\ (2n-1)i - 2j - 8, & \text{if } 2 \leq i \leq k \text{ and } 1 \leq j \leq r-3 \\ (2n-1)i - 2j - 6, & \text{if } 2 \leq i \leq k \text{ and } r-2 \leq j \leq n-7 \\ (2n-1)i - 2n + 5, & \text{if } 2 \leq i \leq k \text{ and } j = n-6, \\ (2n-1)i - 2n + 2j + 1, & \text{if } 2 \leq i \leq k \text{ and } n-5 \leq j \leq n. \end{cases}$$

And, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} n+1, & \text{if } i = 1, \text{ and} j = 1 \\ 2j-2, & \text{if } i = 1, \text{ and } 2 \leq j \leq r+1 \\ 2j-1, & \text{if } i = 1, \text{ and } r+2 \leq j \leq n-1 \\ 3n+4, & \text{if } 1 \leq i \leq k-1 \text{ and } j = n \\ (2n-1)i - 2j - 7, & \text{if } 2 \leq i \leq k \text{ and } 2 \leq j \leq r-3 \\ (2n-1)i - 2j - 13, & \text{if } 2 \leq i \leq k \text{ and } r-2 \leq j \leq n-6 \\ (2n-1)i - n - 4, & \text{if } 2 \leq i \leq k \text{ and } j = n-5 \\ (2n-1)i + 2j - 2n, & \text{if } 2 \leq i \leq k \text{ and } n-4 \leq j \leq n-1 \\ (2n-1)k, & \text{if } i = k \text{ and } j = n. \end{cases}$$

It can be easily proved using mathematical induction on s as in the above theorems that the labeling $f : E(kC_n) \rightarrow \{1, 2, 3, \ldots, (2n-1)k + 1\}$ is an SVM labeling. $\square$

Example 3.2.10. *SVM labeling of a Tridecagonal* (C_{13}) *snake with* 2 *blocks is shown in Figure 3.7.*

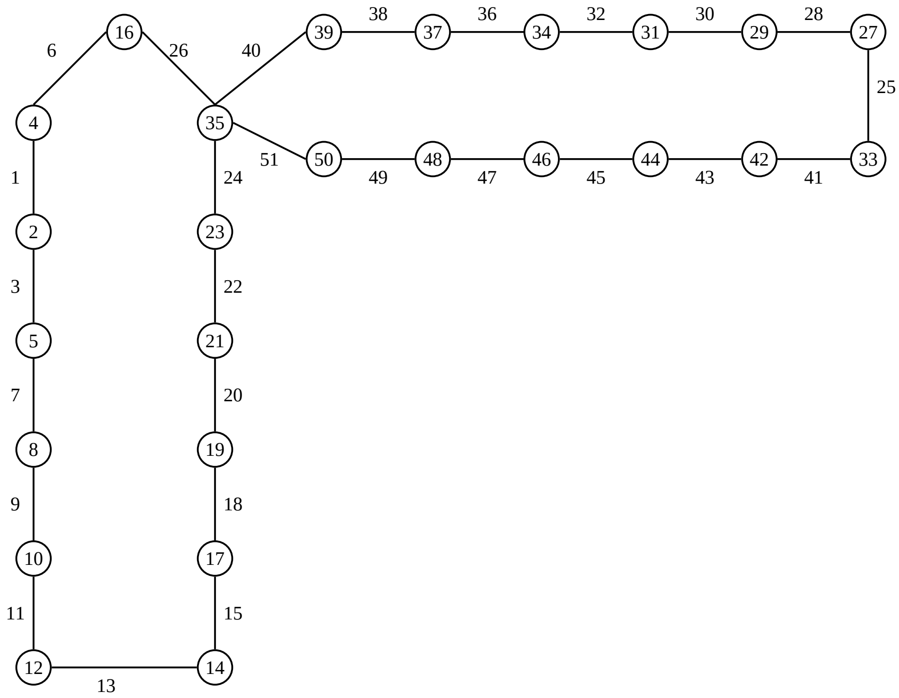


Figure 3.7: SVM labeling of a Tridecagonal (C_{13}) snake with 2 blocks.

3.2.6 kC_n Snakes, $n \geq 10$ and $n \equiv 2(mod\ 4)$

Theorem 3.2.11. *Let* kC_n *be a cyclic snake with* k *blocks of* $C_n, n \geq 10$ *and* $n \equiv 2(mod\ 4)$*. Then* kC_n *is a Super Vertex Mean graph.*

Proof. Let kC_n be a cyclic snake with k blocks of $C_n, n \geq 10$ and $n \equiv 2(mod\ 4)$.
Let $n = 2r$, and $r = 2s + 1$ so that $n = 4s + 2$.

Let $V(kC_n) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq n\}$ and $E(kC_n) = \{e_{i,j} = v_{i,j}v_{i,j+1}\ \&\ e_{i,n} = v_{i,n}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq n-1\}$.

Note that $v_{i,n} = v_{i+1,1}$ for $1 \leq i \leq k-1$, and we refer this vertex as $v_{i,n}$ throughout this proof.

Now, $p = (n-1)k + 1$ and $q = nk$ and $p + q = (2n-1)k + 1$.

Define $f : E(kC_n) \to \{1, 2, 3, \cdots, (2n-1)k+1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 2n-2j-2, & \text{if } i=1, \text{ and } 1 \le j \le r-4 \\ 2n-2j-3, & \text{if } i=1, \text{ and } r-3 \le j \le n-6 \\ 3n-3j-9, & \text{if } i=1, \text{ and } n-5 \le j \le n-4 \\ 1, & \text{if } i=1, \text{ and } j=n-3 \\ 7, & \text{if } i=1, \text{ and } j=n-2 \\ 4n-2j-2, & \text{if } i=1, \text{ and } n-1 \le j \le n \\ 2n-1, & \text{if } i=2, \text{ and } j=1 \\ 2n+2j-2, & \text{if } i=2, \text{ and } 2 \le j \le r-s \\ 2n+2j-1, & \text{if } i=2, \text{ and } r-s+1 \le j \le n-1 \\ (2n-1)i+3, & \text{if } 2 \le i \le k-1 \text{ and } j=n \\ (2n-1)i+2j-2n-1, & \text{if } 3 \le i \le k \text{ and } 1 \le j \le 2 \\ (2n-1)i+2j-2n, & \text{if } 3 \le i \le k \text{ and } 3 \le j \le r-s \\ (2n-1)i+2j-2n+1, & \text{if } 3 \le i \le k \text{ and } r-s+1 \le j \le n-1 \\ (2n-1)k+1, & \text{if } i=k \text{ and } j=n. \end{cases}$$

And, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2n-2j-1, & \text{if } i=1, \text{ and } 1 \le j \le r-3 \\ 2n-2j-2, & \text{if } i=1, \text{ and } r-2 \le j \le n-5 \\ 5, & \text{if } i=1, \text{ and } j=n-4 \\ 8+2j-2n, & \text{if } i=1, \text{ and } n-3 \le j \le n-2 \\ n+4, & \text{if } i=1, \text{ and } j=n-1 \end{cases}$$

$$f^v(v_{i,j}) = \begin{cases} (2n-1)i + r + 1, & \text{if } 1 \leq i \leq k-1 \text{ and } j = n \\ 2n + 2j - 1, & \text{if } i = 2, \text{ and } 2 \leq j \leq r - s \\ 2n + 2j - 2, & \text{if } i = 2, \text{ and } r - s + 1 \leq j \leq n - 1 \\ (2n-1)i + 2n + 2, & \text{if } 3 \leq i \leq k \text{ and } j = 2 \\ (2n-1)i + 2j - 2n - 1, & \text{if } 3 \leq i \leq k \text{ and } 3 \leq j \leq r - s \\ (2n-1)i + 2j - 2n, & \text{if } 3 \leq i \leq k \text{ and } r - s + 1 \leq j \leq n - 1 \\ (2n-1)k, & \text{if } i = k \text{ and } j = n. \end{cases}$$

It can be easily proved using mathematical induction on s as in the above theorems that the labeling $f : E(kC_n) \to \{1, 2, 3, \ldots, (2n-1)k + 1\}$ is an SVM - labeling. □

Example 3.2.12. *Tetradeagonal* (C_{14}) *snake with* 3 *blocks is SVM as shown in Figure* 3.8.

3.3 Conclusion

In this chapter, we have proved that all the cyclic snakes are Super Vertex Mean graphs, provided each s_i on the string $s_1, s_2, s_3, \cdots, s_{k-2}$ which is used to represent a kC_n cycle is equal to 1. This s_i is the distance between i^{th} and $i+1^{th}$ cut vertices along the path, P, where P is the path of minimum length that contains all the cut vertices of a kC_n – snake, starting from one extreme and is taken from $S_n = \{1, 2, 3, \cdots, \lfloor \frac{n}{2} \rfloor\}$.

In the case of Super Mean Labeling, the vertex analogue of SVM, it was easy to obtain a general formula for cyclic snakes represented the string $s_1, s_2, s_3, \cdots, s_{k-2}$, where each s_i need not be equal to 1. This is because when we calculate the induced edge label by finding the average of the labels of two vertices which are the end points of the respective edge, we need to only consider those two vertices. Therefore the average remains the same as in the case of cycles.

But for Super Vertex Mean labeling, when we find the induced vertex labeling of the connecting vertices of a cyclic snake we have to consider four edges that are incident on those vertices to get the average. Thus it becomes pretty difficult to obtain a general formula

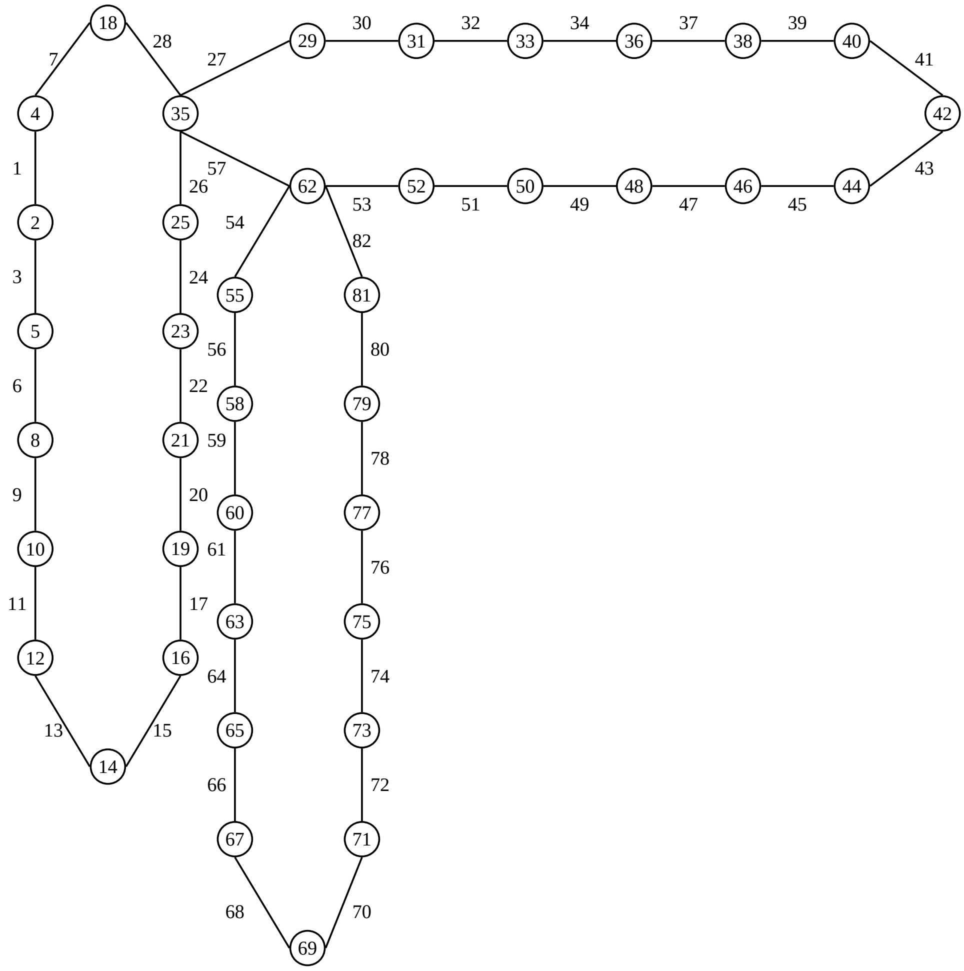


Figure 3.8: SVM labeling of a Tetradecagonal (C_{14}) snake with 3 blocks.

for cyclic snakes represented the string $s_1, s_2, s_3, \cdots, s_{k-2}$, where each s_i need not be equal to 1. Another possibility in this area is to find out SVM - labelings of cyclic graphs whose each s_i is equal, and need not be equal to 1, as we have proved in this paper.

Another possibility that emerges for further study is that we try to explore the SVM - labeling of KC – snakes, which are defined as connecting graphs in which each of the k many blocks is isomorphic to a cycle C_n for some n and the block - cut point graph is a path. As in the case of kC_n - snakes, a kC - snake too can be represented by a string of integers, $s_1, s_2, \cdots, s_{k-2}$. Thus, it is still an open problem to label a kC – snake which has either the same value or different values for each s_i.

Chapter 4

Linear Cyclic Snakes as SVM - Graphs

Linear cyclic snakes are worthy of a special mentioning as the constituent cycles are equally distanced from one another. In the previous chapter we have already defined Cyclic snakes. Here we bring in the slight nuance that is found in the characteristic of linear cyclic snake.

4.1 Linear Cyclic Snakes

Definition 4.1.1. *A kC_n- snake is said to be **linear** if each integer s_i of its string is equal to* $\left\lfloor \frac{n}{2} \right\rfloor$.

Remark: The strings obtained from both the extremes are assumed to be the same. A linear cyclic snake, kC_n is obtained from k copies of C_n by identifying the vertex $v_{i,r+1}$ in the i^{th} copy of C_n at a vertex $v_{i+1,1}$ in the $(i+1)^{th}$ copy of C_n, where $1 \leq i \leq k-1$ and $n = 2r$ or $n = 2r+1$, depending upon whether n is even or odd respectively. We refer this vertex as $v_{i+1,1}$ throughout this chapter.

4.1.1 Known Result

- A linear triangular snake, kC_3 with k blocks is an SVM - graph.

4.2 Linear Cyclic snakes of cycles of higher orders

Now we proceed to prove that other linear cyclic snakes too are Super Vertex Mean Graphs.

4.2.1 Linear Quadrilateral Snake

Theorem 4.2.1. *Linear Quadrilateral snakes, kC_4 with $k \geq 2$ blocks are SVM - graphs.*

Proof. A linear quadratic cyclic snake kC_4 is the graph obtained from $k, k \geq 2$ copies of C_4 by identifying the vertex $v_{i,3}$ in the i^{th} copy of C_4 at a vertex $v_{i+1,1}$ in the $(i+1)^{th}$ copy of C_4, where $1 \leq i \leq k-1$.

Let kC_4 be a linear quadrilateral snake with p vertices and q edges. Then $p = 3k+1$ and $q = 4k$. Suppose we name the vertices of the given linear quadrilateral snake in the anti-clock wise direction, so that

$V(kC_4) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq 4\}$ and

$E(kC_4) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ and $e_{i,4} = v_{i,4}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq 3\}$.

Define $f : E(kC_4) \to \{1, 2, 3, ..., 7k+1\}$ as follows:

When $1 \leq i \leq k-1$, and $k \geq 2$,

$$f(e_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j = 1 \\ 7i-5, & \text{if } 2 \leq i \leq k-1, \text{ and } j = 1 \\ 7i-1, & \text{if } 1 \leq i \leq k-1, \text{ and } j = 2 \\ 7i, & \text{if } 1 \leq i \leq k-1, \text{ and } j = 3 \\ 7i-4, & \text{if } 1 \leq i \leq k-1, \text{ and } j = 4. \end{cases}$$

When $i = k$, $k \geq 2$, and k is even,

$$f(e_{i,j}) = \begin{cases} 7k-6, & \text{if } j = 1, \\ 7k-3, & \text{if } j = 2, \\ 7k-1, & \text{if } j = 3, \\ 7k+1, & \text{if } j = 4. \end{cases}$$

When $i = k$, $k \geq 3$, and k is odd,

$$f(e_{i,j}) = \begin{cases} 7k-5, & \text{if } j = 1, \\ 7k-2, & \text{if } j = 2, \\ 7k+1, & \text{if } j = 3, \\ 7k-4, & \text{if } j = 4. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

When $1 \leq i \leq k-1$, and $k \geq 2$,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 7i-6, & \text{if } 2 \leq i \leq k-1, \text{ and } j = 1, \\ 7i-3, & \text{if } 1 \leq i \leq k-1, \text{ and } j = 2, \\ 7i-2, & \text{if } 1 \leq i \leq k-1, \text{ and } j = 4. \end{cases}$$

When $i = k$, $k \geq 2$, and k is even,

$$f^v(v_{i,j}) = \begin{cases} 7k-5, & \text{if } j = 1, \\ 7k-4, & \text{if } j = 2, \\ 7k-2, & \text{if } j = 3, \\ 7k, & \text{if } j = 4. \end{cases}$$

When $i = k$, $k \geq 3$, and k is odd,

$$f^v(v_{i,j}) = \begin{cases} 7k-6, & \text{if } j = 1, \\ 7k-3, & \text{if } j = 2, \\ 7k, & \text{if } j = 3, \\ 7k-1, & \text{if } j = 4. \end{cases}$$

It can be easily verified that the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 7k+1\}$ as follows;

Case 1: When k is even,

$$f(E) = \{1, 6, 7, 3, 9, 13, 14, 10, 16, 20, 21, 27, \cdots,$$
$$7k-12, 7k-8, 7k-7, 7k-11, 7k-6, 7k-3, 7k-1, 7k+1\}$$

And,

$$f^v(V) = \{2, 4, 8, 5, 11, 15, 12, 18, 22, 19, \cdots,$$
$$7k-10, 7k-5, 7k-9, 7k-4, 7k-2, 7k\}$$

Therefore,

$$f(E) \cup f^v(V) = \{1, 2, 3, 4, \cdots, 7k-12, 7k-11, 7k-10, 7k-9, 7k-8,$$
$$7k-7, 7k-6, 7k-5, 17k-4, 7k-3, 7k-2, 7k-2, 7k, 7k+1\}.$$

Case 2: When k is odd,

$$f(E) = \{1, 6, 7, 3, 9, 13, \cdots, 7k-12, 7k-8, 7k-7,$$
$$7k-11, 7k-5, 7k-2, 7k+1, 7k-4\}$$

And,

$$f^v(V) = \{2, 4, 8, 5, \cdots, 7k-10, 7k-6, 7k-9, 7k-3, 7k, 7k-1\}$$

Therefore,

$$f(E) \cup f^v(V) = \{1, 2, 3, 4, \cdots, 7k-12, 7k-11, 7k-10, 7k-9, 7k-8,$$
$$7k-7, 7k-6, 7k-5, 17k-4, 7k-3, 7k-2, 7k-2, 7k, 7k+1\}.$$

In both the cases above, it has been proved that the labeling $f : E(kC_4) \rightarrow \{1, 2, \cdots, 7k+1\}$ is a Super Vertex Mean labeling. $\square$

Example 4.2.2. *Super vertex-mean labeling of two Linear Quadrilateral snakes with* 4 *and* 3 *blocks are shown in Figures* 4.1 *and* 4.2 *respectively.*

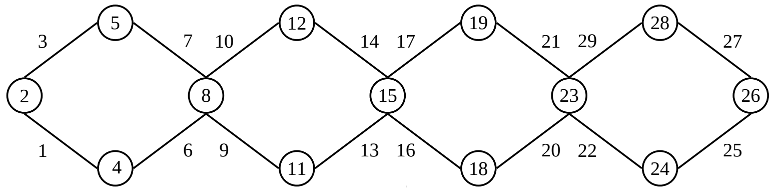


Figure 4.1: A Super vertex-mean labeling of a linear quadrilateral snake with 4 blocks

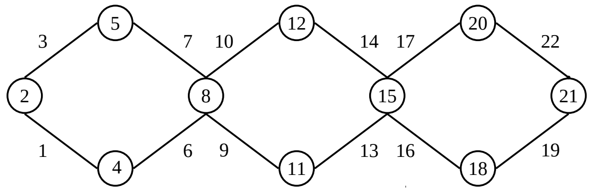


Figure 4.2: Super vertex mean labeling of a linear quadrilateral snake with 3 blocks

4.2.2 Linear Pentagonal Snake

Theorem 4.2.3. *Linear Pentagonal snakes,* kC_5*with* $k, k \geq 2$ *blocks are SVM - graphs.*

Proof. A linear pentagonal cyclic snake kC_5 is the graph obtained from $k, k \geq 2$ copies of C_5 by identifying the vertex $v_{i,3}$ in the i^{th} copy of C_5 at a vertex $v_{i+1,1}$ in the $(i+1)^{th}$ copy of C_5, where $1 \leq i \leq k-1$.

Let kC_5 be a linear pentagonal snake with $k, k \geq 2$ blocks of C_5.

Let,

$V(kC_5) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq 5\}$ and

$E(kC_5) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ and $e_{i,5} = v_{i,5}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq 4\}$.

Now, $p = 4k+1$, $q = 5k$ and $p + q = 9k + 1$.

Define $f : E(kC_5) \to \{1, 2, 3, \cdots, 9k+1\}$ as follows, When $i = 1$,

$$f(e_{i,j}) = \begin{cases} 5, & \text{if } j = 1, \\ 2j+4, & \text{if } 2 \leq j \leq 3, \\ 1, & \text{if } j = 4, \\ 3, & \text{if } j = 5. \end{cases}$$

And When $2 \leq i \leq k$,

$$f(e_{i,j}) = \begin{cases} 9i-9, & \text{if } j = 1, \\ 9i+2j-5, & \text{if } 2 \leq j \leq 3, \\ 9i+3j-18, & \text{if } 4 \leq j \leq 5. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

When $i = 1$,

$$f^v(v_{i,j}) = \begin{cases} 4, & \text{if } j = 1, \\ 7, & \text{if } j = 2, \\ 6, & \text{if } j = 4, \\ 2, & \text{if } j = 5. \end{cases}$$

And when $2 \leq i \leq k$,

$$f^v(v_{i,j}) = \begin{cases} 9i+2j-9, & \text{if } 1 \leq j \leq 2 \\ 9i-2j+6, & \text{if } 4 \leq j \leq 5 \\ 9k, & \text{if } i = k, \text{ and } j = 3. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 9k+1\}$.

Therefore, linear pentagonal snakes kC_5 are Super Vertex Mean graphs. □

Example 4.2.4. *SVM labeling of a linear pentagonal snake with* 3 *blocks is shown in Figure* 4.3.

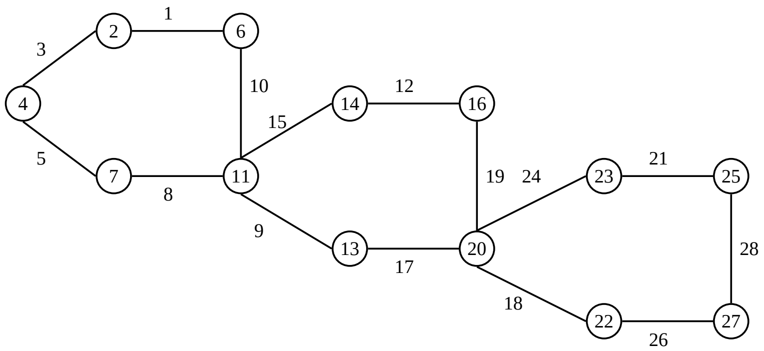


Figure 4.3: SVM labeling of a linear pentagonal snake, $3C_5$.

4.2.3 Linear Hexagonal Snake

Theorem 4.2.5. *Linear Hexagonal snakes, $kC_6, k \geq 2$ are Super Vertex Mean Graphs.*

Proof. Let kC_6 be a hexagonal snake with $k, k \geq 2$ blocks of C_6.

Let,

$V(kC_6) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq 6\}$ and

$E(kC_6) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ and $e_{i,6} = v_{i,6}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq 5\}$.

Now, $p = 5k + 1$ and $q = 6k$ and $p + q = 11k + 1$.

Define $f : E(G_n) \to \{1, 2, 3, \cdots, 11k + 1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 4, \\ 7, & \text{if } i = 1, \text{ and } j = 5, \\ 1, & \text{if } i = 1, \text{ and } j = 6, \\ 11i - 4j, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 2, \\ 11i + 3j - 11, & \text{if } 2 \leq i \leq k, \text{ and } 3 \leq j \leq 4, \\ 11i - 8j + 37, & \text{if } 2 \leq i \leq k, \text{ and } 5 \leq j \leq 6. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 11i + 3j - 12, & \text{if } 1 \le i \le k, \text{ and } 1 \le j \le 2, \\ 8, & \text{if } i = 1, \text{ and } j = 3, \\ 11i - 5, & \text{if } 2 \le i \le k, \text{ and } j = 3, \\ 11k, & \text{if } i = k, \text{ and } j = 4, \\ 11i - 1, & \text{if } 1 \le i \le k, \text{ and } j = 5, \\ 11i - 7, & \text{if } 1 \le i \le k, \text{ and } j = 6. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 11k + 1\}$.

Therefore, linear hexagonal snakes, kC_6 with k blocks of C_6 are Super Vertex Mean graphs. □

Example 4.2.6. *Figure* 4.4 *shows SVM labeling of a linear hexagonal snake with* 3 *blocks.*

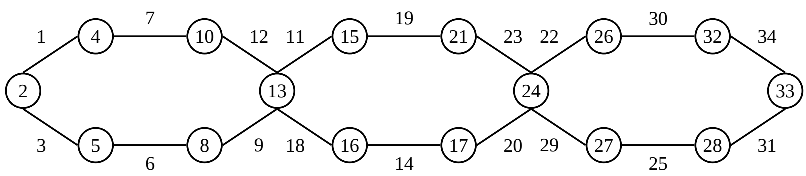


Figure 4.4: A linear hexagonal snake, $3C_6$ is SVM labeled

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4.2.4 Linear Heptagonal Snake

Theorem 4.2.7. *Linear Heptagonal snakes,* $kC_7, k \ge 2$ *are Super Vertex Mean Graphs.*

Proof. Let kC_7 be a linear heptagonal snake with $k, k \ge 2$ blocks of C_7. Let, $V(kC_7) = \{v_{i,j}; 1 \le i \le k, 1 \le j \le 7\}$ and $E(kC_7) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ and $e_{i,7} = v_{i,7}v_{i,1}; 1 \le i \le k, 1 \le j \le 6\}$.

Now, $p = 6k + 1$ and $q = 7k$ and $p + q = 13k + 1$.

Define $f : E(G_n) \to \{1, 2, 3, \cdots, 13k + 1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 4j - 1, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 30 - 4j, & \text{if } i = 1, \text{ and } 4 \leq j \leq 6, \\ 1, & \text{if } i = 1, \text{ and } j = 7, \\ 13i + 2j - 7, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 4, \\ 13i - 13, & \text{if } 2 \leq i \leq k, \text{ and } j = 5, \\ 13i + 2j - 21, & \text{if } 2 \leq i \leq k, \text{ and } 6 \leq j \leq 7. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 4j - 3, & \text{if } i = 1, \text{ and } 2 \leq j \leq 3, \\ 32 - 4j, & \text{if } i = 1, \text{ and } 5 \leq j \leq 7, \\ 13i - 10, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ 13i + 2j - 8, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq 3, \\ 13i - 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 5, \\ 13i + 3j - 29, & \text{if } 2 \leq i \leq k, \text{ and } 6 \leq j \leq 7, \\ 13k, & \text{if } i = k, \text{ and } j = 4. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 13k + 1\}$.

Therefore, linear heptagonal snakes, kC_7 with k blocks of C_7 are Super Vertex Mean graphs. □

Example 4.2.8. *Given in Figure* 4.5 *is an SVM labeling of a linear heptagonal snake,* $3C_7$.

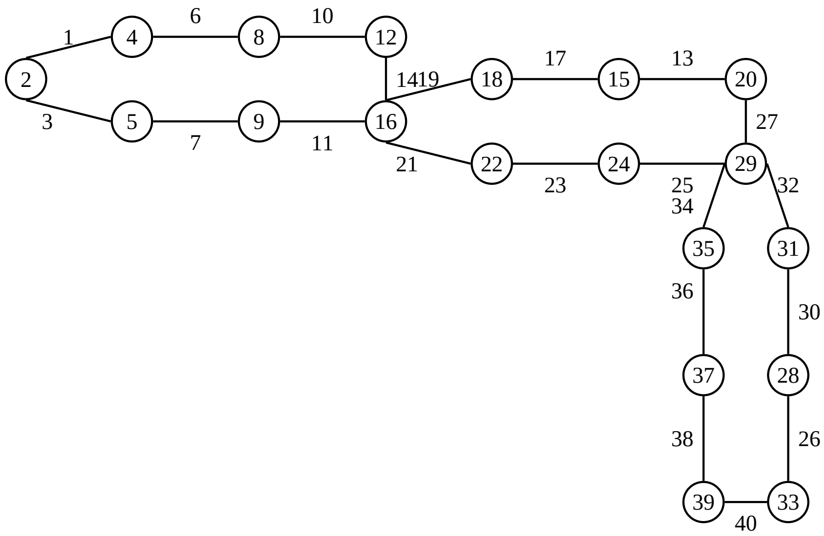


Figure 4.5: Super Vertex Mean Labeling of $3C_7$ linear cyclic snake.

4.2.5 Linear $kC_n, k \geq 2$ blocks of $C_n, n \geq 8$ and $n \equiv 0(mod\, 2)$

Theorem 4.2.9. *Let kC_n be a linear cyclic snake with $k, k \geq 2$ blocks of $C_n, n \geq 8$ and $n \equiv 0(mod\, 2)$. Then kC_n is a Super Vertex Mean graph.*

Proof. Let kC_n be a linear cyclic snake with $k, k \geq 2$ blocks of $C_n, n \geq 8$ and $n \equiv 0(mod\, 2)$ and let $n = 2r, r \geq 4$.

Let $V(kC_n) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq n\}$ and

$E(kC_n) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ & $e_{i,n} = v_{i,n}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq n-1\}$.

Now, $p = (n-1)k + 1$ and $q = nk$ and $p + q = (2n-1)k + 1$.

Define $f : E(kC_n) \rightarrow \{1, 2, 3, \cdots, (2n-1)k + 1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \leq j \leq 3, \\ 4j - 3, & \text{if } i = 1, \text{ and } 4 \leq j \leq r, \\ 4n - 4j + 4, & \text{if } i = 1, \text{ and } r + 1 \leq j \leq n - 2, \\ 7, & \text{if } i = 1, \text{ and } j = n - 1, \\ 1, & \text{if } i = 1, \text{ and } j = n, \end{cases}$$

$$f(e_{i,j}) = \begin{cases} (2n-1)i-2n+8, & \text{if } 2 \leq i \leq k, \text{ and } j=1, \\ (2n-1)i-2n+4, & \text{if } 2 \leq i \leq k, \text{ and } j=2, \\ (2n-1)i-2n+10, & \text{if } 2 \leq i \leq k, \text{ and } j=3, \\ (2n-1)i-2n+4j-2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (2n-1)i+2n-4j+5, & \text{if } 2 \leq i \leq k, \text{ and } r+1 \leq j \leq n-1, \\ (2n-1)(i-1), & \text{if } i=k, \text{ and } j=n. \end{cases}$$

And, the induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i=1, \text{ and } j=1, \\ 3j-1, & \text{if } i=1, \text{ and } 2 \leq j \leq 4, \\ 4j-5, & \text{if } i=1, \text{ and } 5 \leq j \leq r, \\ 4n-4j+6, & \text{if } i=1, \text{ and } r+2 \leq j \leq n-1, \\ 4, & \text{if } i=1, \text{ and } j=n, \\ (2n-1)i-2n+3, & \text{if } 2 \leq i \leq k, \text{ and } j=1, \\ (2n-1)i-2n+6, & \text{if } 2 \leq i \leq k, \text{ and } j=2, \\ (2n-1)i-2n+7, & \text{if } 2 \leq i \leq k, \text{ and } j=3, \\ (2n-1)i-2n+4j-4, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (2n-1)i+2n-4j+7, & \text{if } 2 \leq i \leq k, \text{ and } r+2 \leq j \leq n-1, \\ (2n-1)i-2n+5, & \text{if } 2 \leq i \leq k, \text{ and } j=n, \\ (2n-1)k, & \text{if } i=k, \text{ and } j=r+1. \end{cases}$$

We prove the theorem by mathematical induction on r, where $n = 2r, r \geq 4$.

The above edge labeling function $f(e)$ and the induced vertex labeling function $f^v(v)$ are expressed in terms of r as follows;

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \le j \le 3, \\ 4j - 3, & \text{if } i = 1, \text{ and } 4 \le j \le r, \\ 8r - 4j + 4, & \text{if } i = 1, \text{ and } r + 1 \le j \le 2r - 2, \\ 7, & \text{if } i = 1, \text{ and } j = 2r - 1, \\ 1, & \text{if } i = 1, \text{ and } j = 2r, \\ (4r - 1)i - 4r + 8, & \text{if } 2 \le i \le k, \text{ and } j = 1, \\ (4r - 1)i - 4r + 4, & \text{if } 2 \le i \le k, \text{ and } j = 2, \\ (4r - 1)i - 4r + 10, & \text{if } 2 \le i \le k, \text{ and } j = 3, \\ (4r - 1)i - 4r + 4j - 2, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r, \\ (4r - 1)i + 4r - 4j + 5, & \text{if } 2 \le i \le k, \text{ and } r + 1 \le j \le 2r - 1, \\ (4r - 1)(i - 1), & \text{if } i = k, \text{ and } j = 2r. \end{cases}$$

And the induced vertex labeling in terms of r is,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j - 1, & \text{if } i = 1, \text{ and } 2 \le j \le 4, \\ 4j - 5, & \text{if } i = 1, \text{ and } 5 \le j \le r, \\ 8r - 4j + 6, & \text{if } i = 1, \text{ and } r + 2 \le j \le 2r - 1, \\ 4, & \text{if } i = 1, \text{ and } j = 2r, \\ (4r - 1)i - 4r + 3, & \text{if } 2 \le i \le k, \text{ and } j = 1, \\ (4r - 1)i - 4r + 6, & \text{if } 2 \le i \le k, \text{ and } j = 2, \\ (4r - 1)i - 4r + 7, & \text{if } 2 \le i \le k, \text{ and } j = 3, \end{cases}$$

$$f^v(v_{i,j}) = \begin{cases} (4r-1)i-4r+4j-4, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r, \\ (4r-1)i+4r-4j+7, & \text{if } 2 \le i \le k, \text{ and } r+2 \le j \le 2r-1, \\ (4r-1)i-4r+5, & \text{if } 2 \le i \le k, \text{ and } j = 2r, \\ (4r-1)k, & \text{if } i = k, \text{ and } j = r+1. \end{cases}$$

We prove that the theorem is true when $r = 4$, $n = 8$. When $r = 4$ the linear cyclic snake is a linear octagonal snake with $k, k \ge 2$ cycles of C_8.

Now, $p = 7k + 1$ and $q = 8k$ and $p + q = 15k + 1$.

Define $f : E(kC_n) \to \{1, 2, 3, \cdots, 15k + 1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i = 1, \text{ and } 1 \le j \le 3, \\ 13, & \text{if } i = 1, \text{ and } j = 4, \\ 36-4j, & \text{if } i = 1, \text{ and } 5 \le j \le 6, \\ 7, & \text{if } i = 1, \text{ and } j = 7, \\ 1, & \text{if } i = 1, \text{ and } j = 8, \\ 15i-8, & \text{if } 2 \le i \le k, \text{ and } j = 1, \\ 15i-12, & \text{if } 2 \le i \le k, \text{ and } j = 2, \\ 15i-6, & \text{if } 2 \le i \le k, \text{ and } j = 3, \\ 15i-2, & \text{if } 2 \le i \le k, \text{ and } j = 4, \\ 15i-4j+21, & \text{if } 2 \le i \le k, \text{ and } 5 \le j \le 7, \\ 15i-15, & \text{if } i = k, \text{ and } j = 8. \end{cases}$$

It can be easily verified that f is injective.

The induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j-1, & \text{if } i = 1, \text{ and } 2 \le j \le 4, \\ 38-4j, & \text{if } i = 1, \text{ and } 6 \le j \le 7, \\ 4, & \text{if } i = 1, \text{ and } j = 8, \end{cases}$$

$$f^v(v_{i,j}) = \begin{cases} 15i-13, & \text{if } 2 \le i \le k, \text{ and } j=1, \\ 15i-10, & \text{if } 2 \le i \le k, \text{ and } j=2, \\ 15i-9, & \text{if } 2 \le i \le k, \text{ and } j=3, \\ 15i-4, & \text{if } 2 \le i \le k, \text{ and } j=4, \\ 15i-4j+23, & \text{if } 2 \le i \le k, \text{ and } 6 \le j \le 7, \\ 15i-11, & \text{if } 2 \le i \le k, \text{ and } j=8, \\ 15k, & \text{if } i=k, \text{ and } j=5. \end{cases}$$

Clearly it can be proved that the union of the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 15k+1\}$ as follows;

$$\begin{aligned} f(E) = &\{3, 6, 9, 13, 16, 12, 7, 1\} \cup \\ &\{22, 18, 24, 28, 31, 27, 23, 15\} \cup \cdots, \\ &\{15k-8, 15k-12, 15k-6, 15k-2, 15k+1, \\ &15k-3, 15k-7, 15k-15\}. \\ f^v(V) = &\{2, 5, 8, 11, 14, 10, 4\} \cup \\ &\{17, 20, 21, 26, 29, 25, 19\} \cup \cdots, \\ &\{15k-13, 15k-10, 15k-9, 15k-4, 15k-1, \\ &15k-5, 15k-11, 15k\}. \end{aligned}$$

$$\begin{aligned} f(E) \cup f^v(V) = &\{1, 3, 6, 7, 9, 12, 13, 16\} \cup \{2, 4, 5, 8, 10, 11, 14\} \cup \\ &\{15, 18, 22, 23, 24, 27, 28, 31\} \cup \{17, 19, 20, 21, 25, 26, 29\} \cup \cdots, \\ &\{15k-15, 15k-12, 15k-8, 15k-7, 15k-6, 15k-3, 15k-2, \\ &15k+1\} \cup \{15k-13, 15k-11, 15k-10, 15k-9, 15k-5, \\ &15k-4, 15k-1, 15k\}. \\ &= \{1, 2, 3, 4, \cdots, 29, 30, 31, \cdots, 15k-2, 15k-1, 15k, 15k+1\} \end{aligned}$$

Thus the theorem is true when $r = 4$.

Now we assume that the theorem is true for $r-1, r \geq 5$ (i.e., for $n-2, n \geq 10$).

Now, $p = (n-3)k+1 = (2r-3)k+1$ and $q = (n-2)k = (2r-2)k$ and $p+q = (2n-5)k+1 = (4r-5)k+1$.

The induction hypothesis is that the edge labeling,

$$f : E(kC_{2r-2}) \rightarrow \{1, 2, 3, \cdots, (4r-5)k+1\}$$

defined as follows, is a Super Vertex Mean Labeling, where $r \geq 5, n \geq 10$, $n \equiv 0(mod\ 2)$ and $k \geq 2$.

$$f(e_{i,j}) = \begin{cases} 3j, & \text{if } i=1, \text{ and } 1 \leq j \leq 3, \\ 4j-3, & \text{if } i=1, \text{ and } 4 \leq j \leq r-1, \\ 8r-4j-4, & \text{if } i=1, \text{ and } r \leq j \leq 2r-4, \\ 7, & \text{if } i=1, \text{ and } j=2r-3, \\ 1, & \text{if } i=1, \text{ and } j=2r-2, \\ (4r-5)i-4r+12, & \text{if } 2 \leq i \leq k, \text{ and } j=1, \\ (4r-5)i-4r+8, & \text{if } 2 \leq i \leq k, \text{ and } j=2, \\ (4r-5)i-4r+14, & \text{if } 2 \leq i \leq k, \text{ and } j=3, \\ (4r-5)i-4r+4j+2, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r-1, \\ (4r-5)i+4r-4j+1, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq 2r-3, \\ (4r-5)(i-1), & \text{if } i=k, \text{ and } j=2r-2. \end{cases}$$

And the induced vertex labeling is,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i=1, \text{ and } j=1, \\ 3j-1, & \text{if } i=1, \text{ and } 2 \leq j \leq 4, \\ 4j-5, & \text{if } i=1, \text{ and } 5 \leq j \leq r-1, \\ 8r-4j-2, & \text{if } i=1, \text{ and } r+1 \leq j \leq 2r-3, \end{cases}$$

$$
f^v(v_{i,j}) = \begin{cases}
4, & \text{if } i = 1, \text{ and } j = 2r-2,\\
(4r-5)i - 4r + 7, & \text{if } 2 \le i \le k, \text{ and } j = 1,\\
(4r-5)i - 4r + 10, & \text{if } 2 \le i \le k, \text{ and } j = 2,\\
(4r-5)i - 4r + 11, & \text{if } 2 \le i \le k, \text{ and } j = 3,\\
(4r-5)i - 4r + 4j, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r-1,\\
(4r-5)i + 4r - 4j + 3, & \text{if } 2 \le i \le k, \text{ and } r+1 \le j \le 2r-3,\\
(4r-5)i - 4r + 9, & \text{if } 2 \le i \le k, \text{ and } j = 2r-2,\\
(4r-5)k, & \text{if } i = k, \text{ and } j = r.
\end{cases}
$$

Now we prove that the result is true for any r. If we replace r with $r+1$ in the above mapping we get,

$$
f(e_{i,j}) = \begin{cases}
3j, & \text{if } i = 1, \text{ and } 1 \le j \le 3,\\
4j - 3, & \text{if } i = 1, \text{ and } 4 \le j \le r,\\
8r - 4j + 4, & \text{if } i = 1, \text{ and } r+1 \le j \le 2r-2,\\
7, & \text{if } i = 1, \text{ and } j = 2r-1,\\
1, & \text{if } i = 1, \text{ and } j = 2r,\\
(4r-1)i - 4r + 8, & \text{if } 2 \le i \le k, \text{ and } j = 1,\\
(4r-1)i - 4r + 4, & \text{if } 2 \le i \le k, \text{ and } j = 2,\\
(4r-1)i - 4r + 10, & \text{if } 2 \le i \le k, \text{ and } j = 3,\\
(4r-1)i - 4r + 4j - 2, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r,\\
(4r-1)i + 4r - 4j + 5, & \text{if } 2 \le i \le k, \text{ and } r+1 \le j \le 2r-1,\\
(4r-1)(i-1), & \text{if } i = k, \text{ and } j = 2r.
\end{cases}
$$

And,

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 3j - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq 4, \\ 4j - 5, & \text{if } i = 1, \text{ and } 5 \leq j \leq r, \\ 8r - 4j + 6, & \text{if } i = 1, \text{ and } r + 2 \leq j \leq 2r - 1, \\ 4, & \text{if } i = 1, \text{ and } j = 2r, \\ (4r-1)i - 4r + 3, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (4r-1)i - 4r + 6, & \text{if } 2 \leq i \leq k, \text{ and } j = 2, \\ (4r-1)i - 4r + 7, & \text{if } 2 \leq i \leq k, \text{ and } j = 3, \\ (4r-1)i - 4r + 4j - 4, & \text{if } 2 \leq i \leq k, \text{ and } 4 \leq j \leq r, \\ (4r-1)i + 4r - 4j + 7, & \text{if } 2 \leq i \leq k, \text{ and } r + 2 \leq j \leq 2r - 1, \\ (4r-1)i - 4r + 5, & \text{if } 2 \leq i \leq k, \text{ and } j = 2r, \\ (4r-1)k, & \text{if } i = k, \text{ and } j = r + 1. \end{cases}$$

This is equivalent to the original labeling in terms of n, which is given in the beginning of the proof, and it is clear that $f(E) \cup f^v(V) = \{1, 2, 3, \cdots, (4r-1)k - 1, (4r-1)k, (4r-1)k + 1\}$.

Thus the theorem is proved by Mathematical Induction. □

Example 4.2.10. *Figure* 4.6 *is an SVM - labeling of a linear cyclic snake* $2C_{12}$.

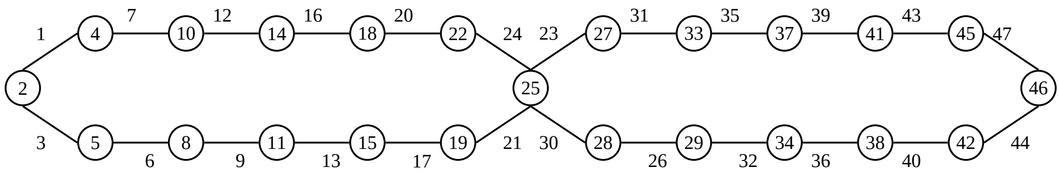


Figure 4.6: SVM - Labeling of a linear cyclic snake, $2C_{12}$.

4.2.6 Linear $kC_n, k \geq 2$ blocks of $C_n, n \geq 9$ and $n \equiv 1(mod\, 4)$

Theorem 4.2.11. *Let kC_n be a linear cyclic snake with $k, k \geq 2$ blocks of $C_n, n \geq 9$ and $n \equiv 1(mod\, 4)$. Then kC_n is a Super Vertex Mean graph.*

Proof. Let kC_n be a cyclic snake with $k, k \geq 2$ blocks of $C_n, n \geq 9$ and $n \equiv 1(mod\, 4)$.
Let $n = 2r + 1, r \geq 4$, and $r = 2s, s \geq 2$ so that $n = 4s + 1$.
Let $V(kC_n) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq n\}$ and
$E(kC_n) = \{e_{i,j} = v_{i,j}v_{i,j+1}\ \&\ e_{i,n} = v_{i,n}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq n-1\}$.
Now, $p = (n-1)k + 1$ and $q = nk$ and $p + q = (2n-1)k + 1$.
Define $f : E(kC_n) \to \{1, 2, 3, \cdots, (2n-1)k + 1\}$ as follows,

$$f(e_{i,j}) = \begin{cases} n, & \text{if } i = 1, \text{ and } j = 1, \\ 2j + n - 1, & \text{if } i = 1, \text{ and } 2 \leq j \leq r+1, \\ 2j - n - 2, & \text{if } i = 1, \text{ and } r+2 \leq j \leq n, \\ (2n-1)i - n - 1, & \text{if } 2 \leq i \leq k, \text{ and } j = 1, \\ (2n-1)i - n + 2j - 2, & \text{if } 2 \leq i \leq k, \text{ and } 2 \leq j \leq r-3, \\ (2n-1)i - n + 2j - 1, & \text{if } 2 \leq i \leq k, \text{ and } r-2 \leq j \leq r-1, \\ (2n-1)i - n + 2j, & \text{if } 2 \leq i \leq k, \text{ and } r \leq j \leq r+1, \\ (2n-1)i - 8, & \text{if } 2 \leq i \leq k, n \neq 9 \text{ and } j = r+2, \\ (2n-1)i - 7, & \text{if } 2 \leq i \leq k, n = 9 \text{ and } j = r+2, \\ (2n-1)i - 2n + 1, & \text{if } 2 \leq i \leq k, \text{ and } j = r+3, \\ (2n-1)i - 3n + 2j - 3, & \text{if } 2 \leq i \leq k, \text{ and } r+4 \leq j \leq n+1-s, \\ (2n-1)i - 3n + 2j - 2, & \text{if } 2 \leq i \leq k, \text{ and } n+2-s \leq j \leq n-1, \\ (2n-1)i - n, & \text{if } 2 \leq i \leq k, \text{ and } j = n. \end{cases}$$

And, the induced vertex labels are as follows:

When $i = 1$, and $n \geq 9$,

$$f^v(v_{i,j}) = \begin{cases} n-1, & \text{if } j = 1, \\ 2j+n-2, & \text{if } 2 \leq j \leq r, \\ n+1, & \text{if } j = r+2, \\ 2j-16, & \text{if } r+3 \leq j \leq n. \end{cases}$$

And when $2 \leq i \leq k$, and $n = 9$,

$$f^v(v_{i,j}) = \begin{cases} 17i-13, & \text{if } j = 1, \\ 17i+3j-14, & \text{if } 2 \leq j \leq 4, \\ 17i, & \text{if } j = 5 \text{ and } i = k, \\ 17i-3, & \text{if } j = 6, \\ 22, & \text{if } j = 7, \\ 19, & \text{if } j = 8, \\ 23, & \text{if } j = 9. \end{cases}$$

And when $2 \leq i \leq k$, and $n \geq 13$,

$$f^v(v_{i,j}) = \begin{cases} (2n-1)i-3r-1, & \text{if } j = 1, \\ (2n-1)i-n+2j-3, & \text{if } 2 \leq j \leq r-3, \\ (2n-1)i-n+2j-2, & \text{if } r-2 \leq j \leq r-1, \\ (2n-1)i-2, & \text{if } j = r, \\ (2n-1)i, & \text{if } i = k, \text{ and } j = r+1, \\ (2n-1)i-3, & \text{if } j = r+2, \\ (2n-1)i-n-3, & \text{if } j = r+3, \\ (2n-1)i-3n+2j-4, & \text{if } r+4 \leq j \leq n+1-s, \\ (2n-1)i-3n+2j-3, & \text{if } n+2-s \leq j \leq n-1, \\ (2n-1)i-n-2, & \text{if } j = n. \end{cases}$$

We can easily prove the theorem by the technique of mathematical induction on s as in the previous theorem. The remaining of the proof is left as an exercise. □

Example 4.2.12. *SVM Labeling of a linear cyclic snake,* $2C_{13}$ *is given in Figure* 4.7*.*

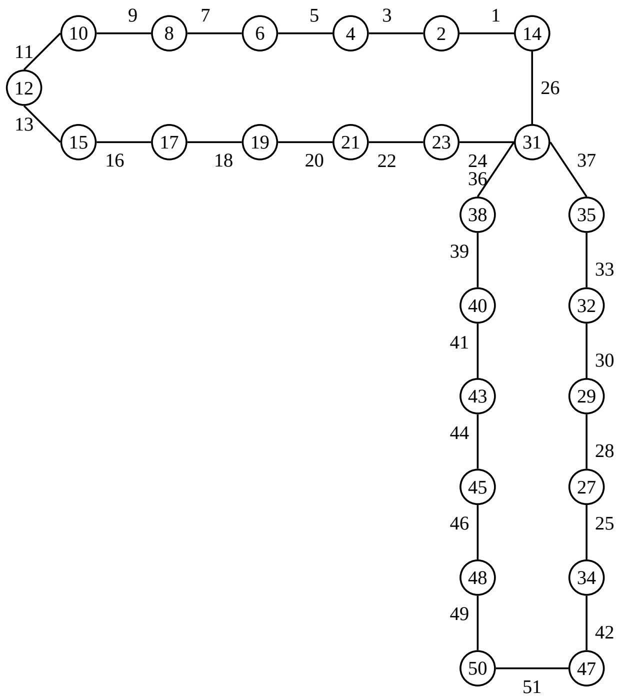


Figure 4.7: SVM - labeling of a linear $2C_{13}$ snake.

4.2.7 Linear $kC_n, k \geq 2$ blocks of $C_n, n \geq 11$ and $n \equiv 3(mod\, 4)$

Theorem 4.2.13. *Let* kC_n *be a linear cyclic snake with* $k, k \geq 2$ *blocks of* $C_n, n \geq 11$ *and* $n \equiv 3(mod\, 4)$*. Then* kC_n *is a Super Vertex Mean graph.*

Proof. Let kC_n be a cyclic snake with $k, k \geq 2$ blocks of $C_n, n \geq 11$ and $n \equiv 3(mod\, 4)$.
Let $n = 2r + 1, r \geq 5$, and $r = 2s + 1, s \geq 2$ so that $n = 4s + 3$.
Let $V(kC_n) = \{v_{i,j}; 1 \leq i \leq k, 1 \leq j \leq n\}$ and
$E(kC_n) = \{e_{i,j} = v_{i,j}v_{i,j+1}$ & $e_{i,n} = v_{i,n}v_{i,1}; 1 \leq i \leq k, 1 \leq j \leq n - 1\}$.
Now, $p = (n - 1)k + 1$ and $q = nk$ and $p + q = (2n - 1)k + 1$.

Define $f : E(kC_n) \to \{1, 2, 3, \cdots, (2n-1)k+1\}$ as follows;

$$f(e_{i,j}) = \begin{cases} 4j-1, & \text{if } i=1, \text{ and } 1 \le j \le r, \\ 4n-4j+2, & \text{if } i=1, \text{ and } r+1 \le j \le 2r, \\ 1, & \text{if } i=1, \text{ and } j=n, \\ (2n-1)i-2n+2j+7, & \text{if } 2 \le i \le k, \text{ and } 1 \le j \le 3, \\ (2n-1)i-2n+4j+1, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r-1, \\ (2n-1)i-3r+3j-2, & \text{if } 2 \le i \le k, \text{ and } r \le j \le r+1, \\ (2n-1)i+2n-4j+2, & \text{if } 2 \le i \le k, \text{ and } r+2 \le j \le 2r-2, \\ (2n-1)i-2n+1, & \text{if } 2 \le i \le k, \text{ and } j=2r-1, \\ (2n-1)i-4n+2j+7, & \text{if } 2 \le i \le k, \text{ and } 2r \le j \le n. \end{cases}$$

And, the induced vertex labels are as follows;

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i=1, \text{ and } j=1, \\ 4j-3, & \text{if } i=1, \text{ and } 2 \le j \le r, \\ 4n-4j+4, & \text{if } i=1, \text{ and } r+2 \le j \le n, \\ (2n-1)i-2n+4, & \text{if } 2 \le i \le k, \text{ and } j=1, \\ (2n-1)i-2n+2j+6, & \text{if } 2 \le i \le k, \text{ and } 2 \le j \le 3, \\ (2n-1)i-2n+4j-1, & \text{if } 2 \le i \le k, \text{ and } 4 \le j \le r, \\ (2n-1)i-1, & \text{if } 2 \le i \le k, \text{ and } j=r+2, \\ (2n-1)i+2n-4j+4, & \text{if } 2 \le i \le k, \text{ and } r+3 \le j \le 2r-2, \\ (2n-1)i-2n+8, & \text{if } 2 \le i \le k, \text{ and } j=2r-1, \\ (2n-1)i-2n+3, & \text{if } 2 \le i \le k, \text{ and } j=2r, \\ (2n-1)i-2n+6, & \text{if } 2 \le i \le k, \text{ and } j=2r+1, \\ (2n-1)k, & \text{if } i=k, \text{ and } j=r+1. \end{cases}$$

We can easily prove that the above labeling is an SVM labeling of kC_n, where $k \geq 2$

blocks of $C_n, n \geq 11$ and $n \equiv 3(mod\, 4)$, by using the technique of mathematical induction on s, where $n = 4s + 3$. Thus the theorem. $\square$

Example 4.2.14. *SVM - Labeling of a linear cyclic snake,* $2C_{15}$ *is given in Figure* 4.8.

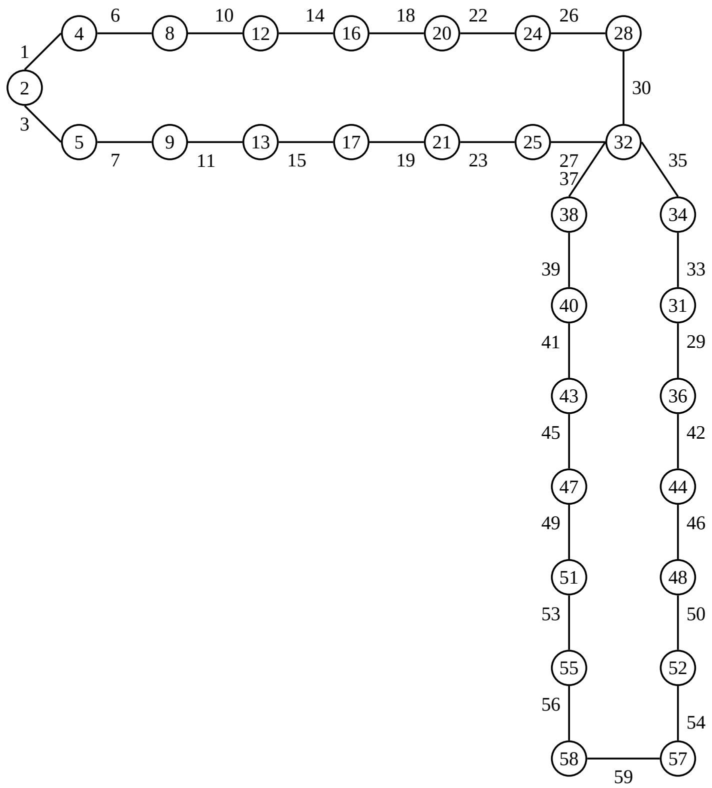


Figure 4.8: SVM - labeling of a linear cyclic snake, $2C_{15}$.

4.3 Conclusion

In this chapter, we have proved that all the linear cyclic snakes are Super Vertex Mean graphs. In the case of Super Mean Labeling, the vertex analogue of SVM, it was easier to obtain a general formula for linear cyclic snakes as well as other cyclic snakes represented by the string $s_1, s_2, s_3, \cdots, s_{k-2}$, where each s_i need not be equal. This is because when we calculate the induced edge label for an edge, by finding the average of the labels of the two vertices which are the end points of that particular edge, we need to consider only those two vertices. Therefore the average remains the same as in the case of cycles.

But for Super Vertex Mean labeling, when we find the induced vertex labeling of the connecting vertices of a cyclic snake we have to consider four edges that are incident on those vertices to get the average. Thus it becomes pretty difficult to obtain a general formula for cyclic snakes represented by the string $s_1, s_2, s_3, \cdots, s_{k-2}$, where each s_i need not be equal. Another possibility emerges is that we try to explore the SVM - labeling of KC – snakes, which is defined as a connecting graph in which each of the k many blocks is isomorphic to a cycle C_n for some n and the block - cut point graph is a path. As in the case of kC_n – snakes, a kC –snake too can be represented by a string of integers, $s_1, s_2, \cdots, s - k - 2$. It remains still an open problem to label a kC –snake which has either equal s_i or different s_i.

Chapter 5

Edge Linked Cyclic Snake as SVM - Graphs

In this chapter we have yet another type of cyclic snakes. This type is an edge analogue of kC_n snake. Whereas in kC_n, each cycle is connected to the next by a means of a vertex, edge linked cyclic snakes are those connected cycles by means of an edge. Here we reproduce the definition and a short discussion on them, from [30] and examine the SVM - behaviour of linear edge linked cyclic ($EL(kC_n)$) snakes.

5.1 Edge Linked Cyclic ($EL(kC_n)$)Snakes

Definition 5.1.1. *A connected graph G obtained from $k, k \geq 2$ copies of a cycle C_n, where $n \geq 4$, by identifying an edge of $(i+1)$th copy, called link i, to an edge of the ith copy for each i, $1 \leq i \leq k-1$, in such a way that consecutive links are not adjacent is called an edge linked cyclic ($EL(kC_n)$) snake.*

5.1.1 Representations of $EL(kC_n)$ - Snake

The way to construct an ($EL(2C_n)$) - snake is unique. For $k \geq 3$, a copy of C_n can be attached in $n-3$ ways to an ($EL(k-1C_n)$) - snake to obtain an ($EL(kC_n)$) - snake. Let G

be and ($EL(kC_n)$) snake. Consider a path P of minimal length that contains all the links of G. Clearly both ends of P are links. Beginning from one of its extreme links, it is possible to construct a string $s'_1, s'_2, s'_3, \cdots, s'_{k-2}$ of $k-2$ integers where the i^{th} integer, s'_i on the string is the number of edges that separates the link- i from the link $(i+1)$ of G on the path P. For each $i, 1 \leq i \leq k-2$, let u_iv_i denote the link i of G on the path P, so that the integer s'_i becomes the length of the $v_i - u_{i+1}$ path on P. As there are $n-3$ different ways to connect the $(i+1)^{th}$ copy of C_n to the i^{th} copy, s'_i is taken from $S'_n = \{1, 2, 3, \cdots, n-3\}$.

Until now, this representation is not unique, because it depends on the extreme of P taken and there are exactly two such paths as P. But, the four strings obtained for both ends of each of the two paths are the same, in the sense that one is obtained from the other by means of one of the following operations;

1. reversing the string
2. replacing each s'_i on the string by $n-2-s'_i$
3. replacing each s'_i on the string by $n-2-s'_i$ and reversing it.

Thus without loss of generality we assume that any ($EL(kC_n)$) - snake is uniquely represented by a string. This is illustrated by the following example.

Example 5.1.2. *An $EL(5C_7)$ - snake represented by its unique string in Figure 5.1.*

Consider graph $G \cong EL(5C_7)$ - snake of figure 4.1. Consider the path $P_1 : abcdefghijklmn$ of minimal length that contains all the links of G. It is clear that ab, de, hi, and mn are links of G. Beginning from the link ab, we observe that two edges separate link 1 (ab) from link 2 (de), thus we have $s'_1 = 2$. In a similar way the integers $s'_2 = 3, s'_3 = 4$ can be obtained. Hence $2, 3, 4$ is a string attached to G. If we had constructed the string beginning from the link 4 (nm), the string would have been $4, 3, 2$. Similarly $3, 2, 1$ and $1, 2, 3$ would have been the strings, had we constructed the string starting from link (ba) or link (mn) respectively using the path $P_2 : barsedtihnm$. It is easy to observe that one string can be obtained from the other by means of any one of the

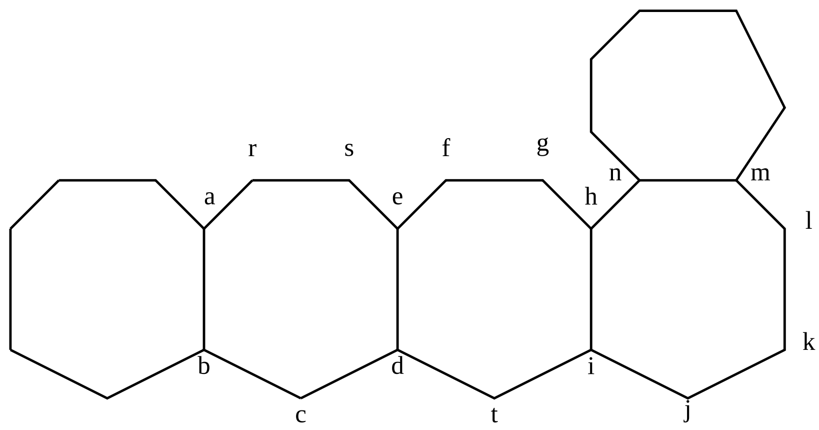


Figure 5.1: An EL$(5C_7)$ - snake represented by its unique string.

operations mentioned earlier, in the construction of the string. Therefore these four strings are considered to be the same and we use any one of them to represent G.

Let G be an $EL(kC_n)$ - snake represented by a string $s'_1, s'_2, s'_3, \cdots, s'_{k-2}$ of $k-2$ integers. Then G is said to be **linear** if $s'_i = \lfloor \frac{n}{2} \rfloor - 1$ or $\lceil \frac{n}{2} \rceil - 1$, for each i.

Remark 5.1.3. *All $EL(kC_n)$ -snakes in this chapter are constructed keeping the following aspects in mind, where $e_{i,j}$ or $v_{i,j}$ represent the j^{th} edge or vertex of the i^{th} cycle in $EL(kC_n)$ respectively, $1 \leq i \leq k$, and $1 \leq j \leq n$;*

1. **When n is odd**

   - $v_{i,\lfloor \frac{n}{2} \rfloor \ or \ \lceil \frac{n}{2} \rceil} = v_{i+1,1}$, where $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ is odd.
   - $v_{i,\lfloor \frac{n}{2} \rfloor+1 \ or \ \lceil \frac{n}{2} \rceil+1} = v_{i+1,n}$, where $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ is odd.
   - $e_{i,\lfloor \frac{n}{2} \rfloor \ or \ \lceil \frac{n}{2} \rceil} = e_{i+1,n}$, where $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ is odd.

2. **When n is even**

   - $v_{i,\frac{n}{2}} = v_{i+1,1}$.
   - $v_{i,\frac{n}{2}+1} = v_{i+1,n}$.
   - $e_{i,\frac{n}{2}} = e_{i+1,n}$.

3. **For all n.**

- $p = (n-2)k+2$, $q = (n-1)k+1$ and $p+q = (2n-3)k+3$.

5.2 Linear $EL(kC_n)$ - snakes and their SVM - Behaviour

From the above discussion it is vividly clear that there is only one $EL(kC_4)$ - snake, the ladder $P_{k+1} \times P_2$, which is an SVM - graph. For the sake of completeness, we begin discussing $EL(kC_4)$ - snakes.

5.2.1 $EL(kC_4)$ - Snake

Theorem 5.2.1. $EL(kC_4)$ - *snake is an SVM graph.*

Proof. Let G be an $EL(kC_4)$ - snake $\cong P_{k+1} \times P_2$. Clearly the order of G is $p = 2k+2$ and the size of G is $q = 3k+1$.
Define $f : E(EL(kC_4)) \to \{1, 2, 3, ..., 5k+3\}$ as follows:

$$f(e_{i,j}) = \begin{cases} 2j+1, & \text{if } i=1, \text{ and } 1 \leq j \leq 3, \\ 1, & \text{if } i=1, \text{ and } j=4, \\ 5i-1, & \text{if } 2 \leq i \leq k-1, \ k \text{ is odd and } j=1, \\ 5i+2j-4, & \text{if } 2 \leq i \leq k-1, \ k \text{ is odd and } 2 \leq j \leq 3, \\ 5i, & \text{if } 2 \leq i \leq k-1, \ k \text{ is even and } j=1, \\ 5i+3j-7, & \text{if } 2 \leq i \leq k-1, \ k \text{ is even and } 2 \leq j \leq 3, \\ 5k+2j-3, & \text{if } i=k, \ 1 \leq j \leq 3. \end{cases}$$

The induced vertex labels are found to be as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 5i + 2j + 3, & \text{if } 1 \leq i \leq k - 1, \ k \text{ is even and } 2 \leq j \leq 3, \\ 4, & \text{if } i = 1, \text{ and } j = 4, \\ 5i + 2j + 2, & \text{if } 1 \leq i \leq k - 1, \ k \text{ is odd and } 2 \leq j \leq 3, \\ 5k + 2j - 4, & \text{if } i = k, \ 2 \leq j \leq 3. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 5k + 3\}$. Therefore $EL(kC_4)$ snake is SVM. □

Example 5.2.2. *Super vertex-mean labeling of a* $EL(5C_4)$ *- snake is shown in Figure* 5.2.

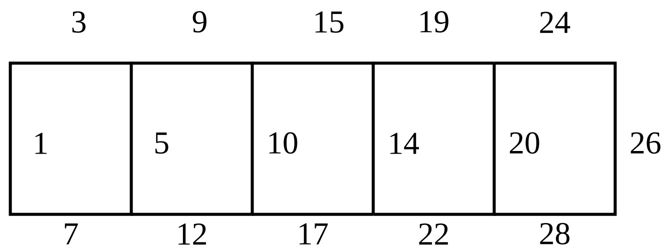


Figure 5.2: EL$(5C_4)$ - snake is SVM graph.

5.2.2 Linear $EL(kC_5)$ - Snake

Theorem 5.2.3. *A Linear* $EL(kC_5)$ *- snake is SVM.*

Proof. Let $EL(kC_5)$ - snake be linear.

Here $p = 3k + 2$, $q = 4k + 1$ and $p + q = 7k + 3$.

Define $f : E(EL(kC_5)) \to \{1, 2, 3, \cdots, 7k+3\}$ as follows;

$$f(e_{i,j}) = \begin{cases} 3, & \text{if } i = 1, \text{ and } j = 1, \\ 7, & \text{if } i = 1, \text{ and } j = 2, \\ 8, & \text{if } i = 1, \text{ and } j = 3, \\ 6, & \text{if } i = 1, \text{ and } j = 4, \\ 1, & \text{if } i = 1, \text{ and } j = 5, \\ 11, & \text{if } i = 2, \text{ and } j = 1, \\ 7i-4, & \text{if } 3 \leq i \leq k, \text{ and } j = 1, \\ 7i-1, & \text{if } 2 \leq i \leq k-1, \text{ and } j = 2, \\ 7i, & \text{if } 2 \leq i \leq k-1, \text{ and } j = 3, \\ 7i+2, & \text{if } 2 \leq i \leq k-1, \text{ and } j = 4, \\ 7k+2j-5, & \text{if } i = k, \text{ and } 2 \leq j \leq 4. \end{cases}$$

The induced vertex labels are found to be as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 5, & \text{if } i = 1, \text{ and } j = 2, \\ 9, & \text{if } i = 1, \text{ and } j = 3, \\ 10, & \text{if } i = 1, \text{ and } j = 4, \\ 4, & \text{if } i = 1, \text{ and } j = 5, \\ 7i+3j-8, & \text{if } 2 \leq i \leq k-1, \text{ and } 2 \leq j \leq 4, \\ 7k+2j-6, & \text{if } i = k, \text{ and } 2 \leq j \leq 4. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 7k+3\}$. Therefore, linear $EL(kC_5)$ - snake is SVM. $\square$

Example 5.2.4. *Figure* 5.3 *is an SVM labeling of a linear* $EL(4C_5)$ *- snake.*

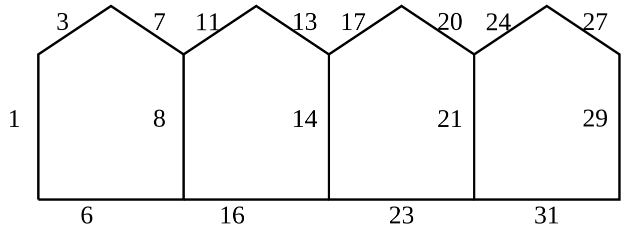


Figure 5.3: SVM labeling of a linear EL$(4C_5)$ - snake.

5.2.3 Linear $EL(kC_6)$ - Snake

Theorem 5.2.5. *Linear $EL(kC_6)$ - snake is a Super Vertex Mean Graph.*

Proof. Let $EL(kC_6)$ be a linear edge linked cyclic snake. Then $p + q = 9k + 3$. Define $f : E(EL(kC_6)) \to \{1, 2, 3, \cdots, 9k + 3\}$ as follows;

$$f(e_{i,j}) = \begin{cases} 2j+1, & \text{if } i=1, \text{ and } 1 \le j \le 5, \\ 1, & \text{if } i=1, \text{ and } j=6, \\ 9i+2j-7, & \text{if } 2 \le i \le k-1, \text{ and } 1 \le j \le 2, \\ 9i-2, & \text{if } 2 \le i \le k-1, \text{ and } j=3, \\ 9i+3j-13, & \text{if } 2 \le i \le k-1, \text{ and } 4 \le j \le 5, \\ 9k+2j-7, & \text{if } i=k, \text{ and } 1 \le j \le 5. \end{cases}$$

The induced vertex labels are found to be as follows:

$$f^v(v_{i,j}) = \begin{cases} 2j, & \text{if } i=1, \text{ and } 1 \le j \le 2, \\ 8, & \text{if } i=1, \text{ and } j=3, \\ 20-2j, & \text{if } i=1, \text{ and } 4 \le j \le 5, \\ 6, & \text{if } i=1, \text{ and } j=6, \\ 9i+4j-12, & \text{if } 2 \le i \le k-1, \text{ and } 2 \le j \le 3, \\ 9i-2j+11, & \text{if } 2 \le i \le k-1, \text{ and } 4 \le j \le 5, \\ 9k+2j-8, & \text{if } i=k, \text{ and } 2 \le j \le 5. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 9k + 3\}$. Therefore, linear $EL(kC_6)$ - snake is SVM. □

Example 5.2.6. *Figure* 5.4 *shows SVM - labeling of a linear* $EL(3C_6)$.

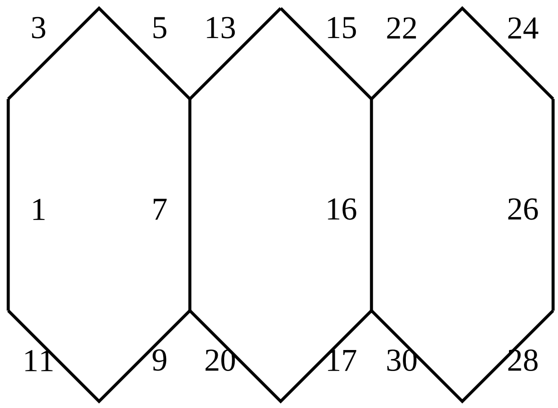


Figure 5.4: SVM - labeling of a linear EL($3C_6$).

5.2.4 Linear $EL(kC_7)$ - Snake

Theorem 5.2.7. *Linear* $EL(kC_7)$*- snake is a Super Vertex Mean Graph.*

Proof. Let $EL(kC_7)$ be a linear. $p + q = 11k + 3$.

Define $f : E(EL(kC_7)) \to \{1, 2, 3, \cdots, 11k + 3\}$ as follows;

$$f(e_{i,j}) = \begin{cases} 3, & \text{if } i = 1, \text{ and } j = 1, \\ 7, & \text{if } i = 1, \text{ and } j = 2, \\ 19 - 2j, & \text{if } i = 1, \text{ and } 3 \leq j \leq 5, \\ 6, & \text{if } i = 1, \text{ and } j = 6, \\ 1, & \text{if } i = 1, \text{ and } j = 7, \\ 11i + 2j - 9, & \text{if } 2 \leq i \leq k, \text{ and } 1 \leq j \leq 2, \end{cases}$$

$$f(e_{i,j}) = \begin{cases} 11i - 2j + 8, & \text{if } 2 \leq i \leq k-1, \text{ and } 3 \leq j \leq 6, \\ 11i - 2j + 9, & \text{if } i = k, \text{ and } 3 \leq j \leq 4, \\ 11i - 2j + 8, & \text{if } i = k, \text{ and } 5 \leq j \leq 6. \end{cases}$$

The induced vertex labels are found to be as follows:

$$f^v(v_{i,j}) = \begin{cases} 2, & \text{if } i = 1, \text{ and } j = 1, \\ 5, & \text{if } i = 1, \text{ and } j = 2, \\ 6 + 2j, & \text{if } i = 1, \text{ and } 3 \leq j \leq 4, \\ 20 - 2j, & \text{if } i = 1, \text{ and } 5 \leq j \leq 6, \\ 4, & \text{if } i = 1, \text{ and } j = 7, \\ 11i - 6, & \text{if } 2 \leq i \leq k-1, \text{ and } j = 2, \\ 11i + 2j - 5, & \text{if } 2 \leq i \leq k-1, \text{ and } 3 \leq j \leq 4, \\ 11i - 2j + 9, & \text{if } 2 \leq i \leq k-1, \text{ and } 5 \leq j \leq 6, \\ 11k - 6, & \text{if } i = k, \text{ and } j = 2, \\ 11k - 1, & \text{if } i = k, \text{ and } j = 3, \\ 11k - 2j + 10, & \text{if } i = k, \text{ and } 4 \leq j \leq 5, \\ 11k - 3, & \text{if } i = k, \text{ and } j = 6. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 11k + 3\}$. Therefore linear $EL(kC_7)$ - snake is SVM. □

Example 5.2.8. *Given in Figure 5.5 is an SVM - labeling of a linear $EL(3C_7)$.*

5.2.5 Linear $EL(kC_8)$ - Snake

Theorem 5.2.9. *Linear $EL(kC_8)$ is a Super Vertex Mean graph.*

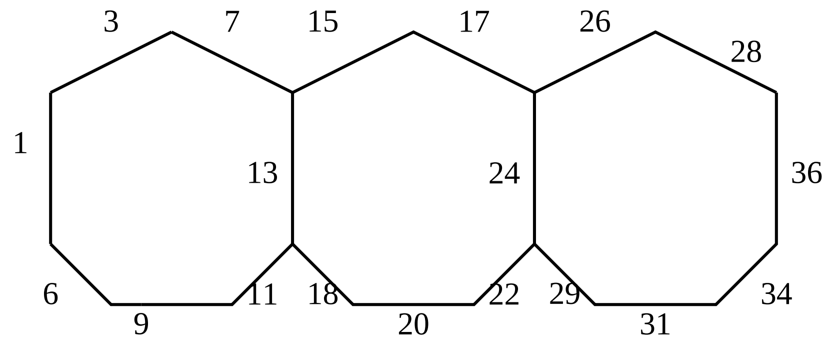


Figure 5.5: SVM - labeling of a linear EL($3C_7$).

Proof. Let $EL(kC_8)$ be a linear edge linked cyclic snake. Then $p+q=13k+3$. Define $f: E(EL(kC_8)) \to \{1,2,3,\cdots,13k+3\}$ as follows;

$$f(e_{i,j}) = \begin{cases} 2j+1, & \text{if } i=1, \text{ and } 1 \le j \le 3, \\ 5+j, & \text{if } i=1, \text{ and } 4 \le j \le 5, \\ 2j+1, & \text{if } i=1, \text{ and } 6 \le j \le 7, \\ 1, & \text{if } i=1, \text{ and } j=8, \\ 13i+2j-11, & \text{if } 2 \le i \le k-1, \text{ and } 1 \le j \le 3, \\ 13i+j-8, & \text{if } 2 \le i \le k-1, \text{ and } 4 \le j \le 5, \\ 13i+2j-12, & \text{if } 2 \le i \le k-1, \text{ and } 6 \le j \le 7, \\ 13k+2j-11, & \text{if } i=k, \text{ and } 1 \le j \le 7. \end{cases}$$

The induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2j, & \text{if } i=1, \text{ and } 1 \le j \le 3, \\ 5j-9, & \text{if } i=1, \text{ and } 4 \le j \le 5, \\ 2j, & \text{if } i=1, \text{ and } 6 \le j \le 7, \\ 8, & \text{if } i=1, \text{ and } j=8, \end{cases}$$

$$f^v(v_{i,j}) = \begin{cases} 13i + 2j - 12, & \text{if } 2 \leq i \leq k-1, \text{ and } 2 \leq j \leq 3, \\ 13i + 5j - 22, & \text{if } 2 \leq i \leq k-1, \text{ and } 4 \leq j \leq 5, \\ 13i + 2j - 13, & \text{if } 2 \leq i \leq k-1, \text{ and } 6 \leq j \leq 7, \\ 13k + 2j - 12, & \text{if } i = k, \text{ and } 2 \leq j \leq 7. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 13k + 3\}$. Therefore linear $EL(kC_8)$ - snake is SVM. □

Example 5.2.10. *Figure* 5.6 *gives SVM - labeling of a linear* $EL(4C_8)$.

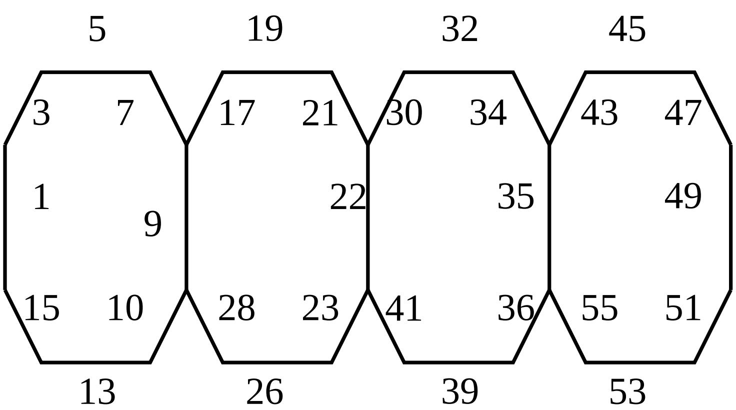


Figure 5.6: SVM - labeling of a linear EL($4C_8$).

5.2.6 Linear $EL(kC_9)$ - Snake

Theorem 5.2.11. *A linear* $EL(kC_9)$ *is SVM.*

Proof. Let $EL(kC_9)$ be a linear edge linked cyclic snake. Then $p + q = 15k + 3$.

Define $f : E(EL(kC_9)) \to \{1, 2, 3, \cdots, 15k + 3\}$ as follows;

$$f(e_{i,j}) = \begin{cases} 2j-1, & \text{if } i=1, \text{ and } 1 \le j \le 2, \\ 6, & \text{if } i=1, \text{ and } j=3, \\ 2j+2, & \text{if } i=1, \text{ and } 4 \le j \le 5, \\ 29-2j, & \text{if } i=1, \text{ and } 6 \le j \le 7, \\ 11, & \text{if } i=1, \text{and } j=8, \\ 7, & \text{if } i=1, \text{ and } j=9, \\ 15i+2j-13, & \text{if } 2 \le i \le k-1, \text{ and } 1 \le j \le 5, \\ 15i-2j+14, & \text{if } 2 \le i \le k-1, \text{ and } 6 \le j \le 7, \\ 15i-4, & \text{if } 2 \le i \le k-1, \text{ and } j=8, \\ 15k+2j-13, & \text{if } i=k, \text{ and } 1 \le j \le 4, \\ 15k-2j+13, & \text{if } i=k, \text{ and } 5 \le j \le 6, \\ 15k-2j+12, & \text{if } i=k, \text{ and } 7 \le j \le 8. \end{cases}$$

The induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 4, & \text{if } i=1, \text{ and } j=1, \\ 3j-4, & \text{if } i=1, \text{ and } 2 \le j \le 4, \\ 4j-6, & \text{if } i=1, \text{ and } 5 \le j \le 6, \\ 37-3j, & \text{if } i=1, \text{ and } 7 \le j \le 8, \\ 9, & \text{if } i=1, \text{ and } j=9, \\ 15i+2j-14, & \text{if } 2 \le i \le k-1, \text{ and } 2 \le j \le 4, \\ 15i+4j-21, & \text{if } 2 \le i \le k-1, \text{ and } 5 \le j \le 6, \\ 15i-3j+22, & \text{if } 2 \le i \le k-1, \text{ and } 7 \le j \le 8, \end{cases}$$

$$f^v(v_{i,j}) = \begin{cases} 15k+2j-14, & \text{if } i=k, \text{ and } 2 \leq j \leq 4, \\ 15k+3j-16, & \text{if } i=k, \text{ and } 5 \leq j \leq 6, \\ 15k-3j+21, & \text{if } i=k, \text{ and } 7 \leq j \leq 8. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 13k+3\}$. Therefore linear $EL(kC_9)$ - snake is SVM. □

Example 5.2.12. *Figure* 5.7 *shows SVM - labeling of a linear* $EL(3C_9)$.

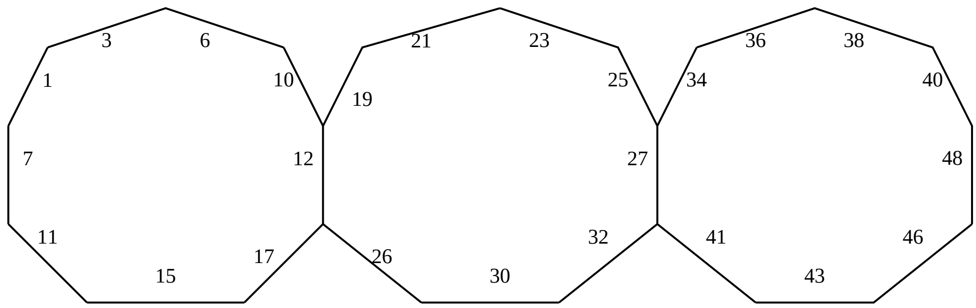


Figure 5.7: SVM - labeling of a linear EL$(3C_9)$.

5.2.7 Linear $EL(kC_{10})$ - Snake

Theorem 5.2.13. *A linear* $EL(kC_{10})$ *is SVM.*

Proof. Let $EL(kC_{10})$ be a linear edge linked cyclic snake. Then $p+q = 17k+3$. Define $f : E(EL(kC_{10})) \rightarrow \{1, 2, 3, \cdots, 17k+3\}$ as follows;

$$f(e_{i,j}) = \begin{cases} 2j+1, & \text{if } i=1, \text{ and } 1 \le j \le 2, \\ 2j+2, & \text{if } i=1, \text{ and } 3 \le j \le 5, \\ 31-2j, & \text{if } i=1, \text{ and } 6 \le j \le 8, \\ 11, & \text{if } i=1, \text{ and } j=9, \\ 1, & \text{if } i=1, \text{ and } j=10, \\ 17i+2j-15, & \text{if } 2 \le i \le k-1, \text{ and } 1 \le j \le 5, \\ 17i-2j+14, & \text{if } 2 \le i \le k-1, \text{ and } 6 \le j \le 8, \\ 17i-6, & \text{if } 2 \le i \le k-1, \text{ and } j=9, \\ 17k+2j-15, & \text{if} i=k, \text{ and } 1 \le j \le 5, \\ 17k-2j+15, & \text{if } i=k, \text{ and } 6 \le j \le 7, \\ 17k-2j+14, & \text{if } i=k, \text{ and } 8 \le j \le 9. \end{cases}$$

The induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 2j, & \text{if } i=1, \text{ and } 1 \le j \le 2, \\ 2j+1, & \text{if } i=1, \text{ and } 3 \le j \le 4, \\ 6j-16, & \text{if } i=1, \text{ and } 5 \le j \le 6, \\ 32-2j, & \text{if } i=1, \text{ and } 7 \le j \le 8, \\ 13, & \text{if } i=1, \text{ and } j=9, \\ 6, & \text{if } i=1, \text{ and } j=10, \\ 17i-12, & \text{if } 2 \le i \le k-1, \text{ and } j=2, \\ 17i+2j-16, & \text{if } 2 \le i \le k-1, \text{ and } 3 \le j \le 4, \\ 17i+6j-33, & \text{if } 2 \le i \le k-1, \text{ and } 5 \le j \le 6, \\ 17i-2j+15, & \text{if } 2 \le i \le k-1, \text{ and } 7 \le j \le 8, \\ 17i-4, & \text{if } 2 \le i \le k-1, \text{ and } j=9, \end{cases}$$

$$
f^v(v_{i,j}) = \begin{cases} 17k-12, & \text{if } i=k, \text{ and } j=2, \\ 17k+2j-16, & \text{if } i=k, \text{ and } 3 \leq j \leq 4, \\ 17k-6, & \text{if } i=k, \text{ and } j=5, \\ 17k+3j-19, & \text{if } i=k, \text{ and } 6 \leq j \leq 7, \\ 17k-3j+24, & \text{if } i=k, \text{ and } 8 \leq j \leq 9. \end{cases}
$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1,2,3,...,17k+3\}$. Therefore linear $EL(kC_{10})$ - snake is SVM. □

Example 5.2.14. *Figure* 5.8 *shows SVM - labeling of a linear* $EL(3C_{10})$.

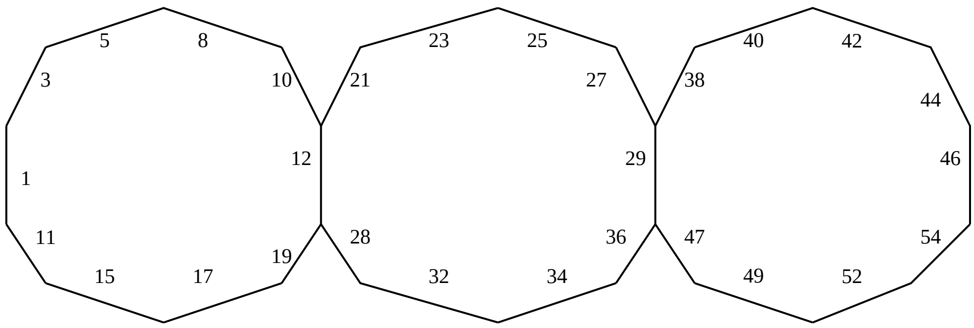


Figure 5.8: SVM - labeling of a linear EL$(3C_{10})$.

5.2.8 Linear $EL(kC_{11})$ - Snake

Theorem 5.2.15. *A linear* $EL(kC_{11})$ *is SVM.*

Proof. Let $EL(kC_{11})$ be a linear edge linked cyclic snake. Then $p+q=19k+3$.

Define $f : E(EL(kC_{11})) \rightarrow \{1, 2, 3, \cdots, 19k+3\}$ as follows;

$$f(e_{i,j}) = \begin{cases} 2j-1, & \text{if } i=1, \text{ and } 1 \leq j \leq 2, \\ 4j-6, & \text{if } i=1, \text{ and } 3 \leq j \leq 4, \\ 14, & \text{if } i=1, \text{ and } j=5, \\ 33-2j, & \text{if } i=1, \text{ and } 6 \leq j \leq 8, \\ 31-2j, & \text{if } i=1, \text{ and } 9 \leq j \leq 10, \\ 7, & \text{if } i=1, \text{ and } j=11, \\ 19i+2j-17, & \text{if } 2 \leq i \leq k-1, \text{ and } 1 \leq j \leq 4, \\ 19i-5, & \text{if } 2 \leq i \leq k-1, \text{ and } j=5, \\ 19i-2j+14, & \text{if } 2 \leq i \leq k-1, \text{ and } 6 \leq j \leq 8, \\ 19i-2j+12, & \text{if } 2 \leq i \leq k-1, \text{ and } 9 \leq j \leq 10, \\ 19k+2j-17, & \text{if } i=k, \text{ and } 1 \leq j \leq 4, \\ 19k-2j+13, & \text{if } i=k, \text{ and } 5 \leq j \leq 7, \\ 19k-2j+12, & \text{if } i=k, \text{ and } 8 \leq j \leq 10. \end{cases}$$

The induced vertex labels are as follows:

$$f^v(v_{i,j}) = \begin{cases} 6-2j, & \text{if } i=1, \text{ and } 1 \leq j \leq 2, \\ 3j-4, & \text{if } i=1, \text{ and } 3 \leq j \leq 4, \\ 16, & \text{if } i=1, \text{ and } j=5, \\ 34-2j, & \text{if } i=1, \text{ and } 6 \leq j \leq 8, \\ 42-3j, & \text{if } i=1, \text{ and } 9 \leq j \leq 10, \\ 9, & \text{if } i=1, \text{ and } j=11, \\ 19i+2j-18, & \text{if } 2 \leq i \leq k-1, \text{ and } 2 \leq j \leq 4, \\ 19i-3, & \text{if } 2 \leq i \leq k-1, \text{ and } j=5, \end{cases}$$

$$f^v(v_{i,j}) = \begin{cases} 19i - 2j + 15, & \text{if } 2 \leq i \leq k-1, \text{ and } 6 \leq j \leq 8, \\ 19i - 3j + 23, & \text{if } 2 \leq i \leq k-1, \text{ and } 9 \leq j \leq 10, \\ 19k + 2j - 18, & \text{if } i = k, \text{ and } 2 \leq j \leq 4, \\ 19k - 3, & \text{if } i = k, \text{ and } j = 5, \\ 19k - 2j + 14, & \text{if } i = k, \text{ and } 6 \leq j \leq 8, \\ 19k - 2j + 13, & \text{if } i = k, \text{ and } 9 \leq j \leq 10. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., 19k + 3\}$. Therefore linear $EL(kC_{11})$ snake is SVM. □

Example 5.2.16. *Figure* 5.9 *gives SVM - labeling of a linear* $EL(3C_{11})$.

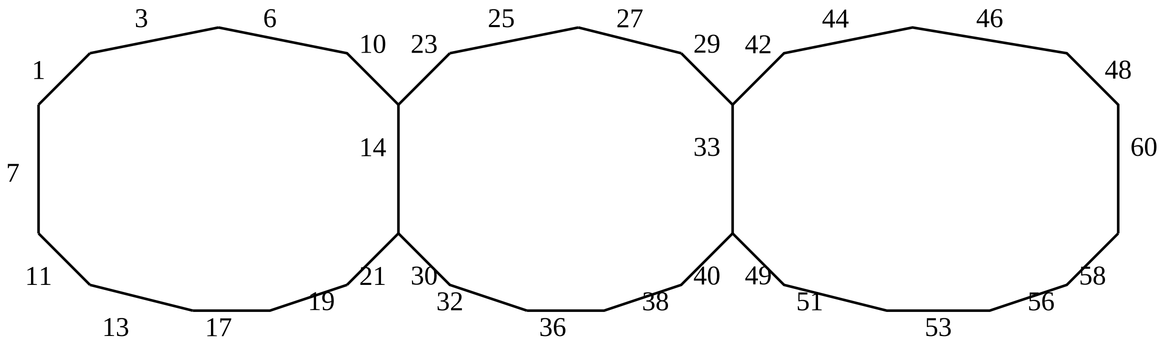


Figure 5.9: SVM - labeling of a linear EL$(3C_{11})$.

5.3 Linear Edge Linked Snakes of Higher Orders

5.3.1 Linear $EL(kC_n)$ - Snake, $n \equiv 0(mod\ 12)$ and $n \geq 12$

Theorem 5.3.1. *Let* $EL(kC_n)$ *be a linear edge linked cyclic snake, where* $n \equiv 0(mod\ 12)$ *and* $n \geq 12$. *Then* $EL(kC_n)$ *is SVM.*

Proof. Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 0(mod\, 12)$ and $n \geq 12$. We know that $p + q = (2n-3)k + 3$.

Define $f : E(EL(kC_n)) \to \{1, 2, 3, \cdots, (2n-3)k + 3\}$ as follows;

Case 1a: For $i = 1$

$$f(e_{i,j}) = \begin{cases} n - 2j - 1, & \text{if } 1 \leq j \leq \frac{n}{2} - 6, \\ 2j - n + 11, & \text{if } \frac{n}{2} - 5 \leq j \leq \frac{n}{2} - 3, \\ 2j - n + 12, & \text{if } \frac{n}{2} - 2 \leq j \leq \frac{n}{2} - 1, \\ n + 2, & \text{if } j = \frac{n}{2}, \\ 3n - 2j + 1, & \text{if } \frac{n}{2} + 1 \leq j \leq n - 2, \\ 3n - 2j - 1, & \text{if } n - 1 \leq j \leq n. \end{cases}$$

Case 2a: For $2 \leq i \leq k - 1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \frac{n}{2} - 1, \\ (2n-3)i - n + 5, & \text{if } j = \frac{n}{2}, \\ (2n-3)i + n - 2j + 4, & \text{if } \frac{n}{2} + 1 \leq j \leq \frac{5}{6}n - 1, \\ (2n-3)i + n - 2j + 3, & \text{if } \frac{5}{6}n \leq j \leq n - 2, \\ (2n-3)i - n + 4, & \text{if } j = n - 1. \end{cases}$$

Case 3a: For $i = k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \frac{n}{2} - 1, \\ (2n-3)i + n - 2j + 3, & \text{if } \frac{n}{2} \leq j \leq \frac{3}{4}n - 1, \\ (2n-3)i + n - 2j + 2, & \text{if } \frac{3}{4}n \leq j \leq n - 1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} n-2j, & \text{if } 1 \le j \le \frac{n}{2}-6, \\ 6, & \text{if } j = \frac{n}{2}-5, \\ 2j-n+10, & \text{if } \frac{n}{2}-4 \le j \le \frac{n}{2}-3, \\ 2j-n+11, & \text{if } \frac{n}{2}-2 \le j \le \frac{n}{2}-1, \\ n+4, & \text{if } j = \frac{n}{2}, \\ 3n-2j+2, & \text{if } \frac{n}{2}+1 \le j \le n-2, \\ n+3, & \text{if } j = n-1, \\ n, & \text{if } j = n. \end{cases}$$

Case 2b: For $2 \le i \le k-1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i-2n+2j+4, & \text{if } 2 \le j \le \frac{n}{2}-1, \\ (2n-3)i-\frac{2}{3}n+4, & \text{if } j = \frac{n}{2}, \\ (2n-3)i+n-2j+5, & \text{if } \frac{n}{2}+1 \le j \le \frac{5}{6}n, \\ (2n-3)i+n-2j+4, & \text{if } \frac{5}{6}n+1 \le j \le n-1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i-2n+2j+4, & \text{if } 2 \le j \le \frac{n}{2}-1, \\ (2n-3)i-\frac{n}{2}+3, & \text{if } j = \frac{n}{2}, \\ (2n-3)i+n-2j+4, & \text{if } \frac{n}{2}+1 \le j \le \frac{3}{4}n, \\ (2n-3)i+n-2j+3, & \text{if } \frac{3}{4}n+1 \le j \le n-1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k+3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 0(mod\ 12)$ and $n \ge 12$, is SVM. □

5.3.2 Linear $EL(kC_n)$ - Snake, $n \equiv 1(mod\,12)$ and $n \geq 13$

Theorem 5.3.2. *Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 1(mod\,12)$ and $n \geq 13$. Then EL(kC_n) is SVM.*

Proof. Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 1(mod\,12)$ and $n \geq 13$. Then $p+q=(2n-3)k+3$.

Define $f: E(EL(kC_n)) \to \{1,2,3,\cdots,(2n-3)k+3\}$ as follows;

Case 1a: For $i=1$

$$f(e_{i,j}) = \begin{cases} n-2j, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 7, \\ 7, & \text{if } j = \lceil \frac{n}{2} \rceil - 6, \\ 1, & \text{if } j = \lceil \frac{n}{2} \rceil - 5, \\ 3j - 3\lceil \frac{n}{2} \rceil + 15, & \text{if } \lceil \frac{n}{2} \rceil - 4 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ n+3, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ 3n-2j+2, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq n-2, \\ 3n-2j, & \text{if } n-1 \leq j \leq n. \end{cases}$$

Case 2a: For $2 \leq i \leq k-1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i-2n+2j+5, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i-n+6, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ (2n-3)i+n-2j+5, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq \lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i+n-2j+4, & \text{if } \lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{2} \rceil \leq j \leq n-2, \\ (2n-3)i-n+5, & \text{if } j = n-1. \end{cases}$$

Case 3a: For $i=k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i-2n+2j+5, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i+n-2j+4, & \text{if } \lceil \frac{n}{2} \rceil \leq j \leq \frac{1}{2}(n+\lceil \frac{n}{2} \rceil) - 1, \\ (2n-3)i+n-2j+3, & \text{if } \frac{1}{2}(n+\lceil \frac{n}{2} \rceil) \leq j \leq n-1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} n-2j+1, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 7, \\ 10, & \text{if } j = \lceil \frac{n}{2} \rceil - 6, \\ 4, & \text{if } j = \lceil \frac{n}{2} \rceil - 5, \\ 3j - 3\lceil \frac{n}{2} \rceil + 14, & \text{if } \lceil \frac{n}{2} \rceil - 4 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ n+5, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ 3n-2j+3, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq n-2, \\ 4n-3j+1, & \text{if } n-1 \leq j \leq n. \end{cases}$$

Case 2b: For $2 \leq i \leq k-1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i - 2\lfloor \frac{n}{3} \rfloor + 4, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ (2n-3)i + n - 2j + 6, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq \lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{2} \rceil, \\ (2n-3)i + n - 2j + 5, & \text{if } \lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{2} \rceil + 1 \leq j \leq n-1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i - \lceil \frac{n}{2} \rceil + 4, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ (2n-3)i + n - 2j + 5, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq \frac{1}{2}(n + \lceil \frac{n}{2} \rceil), \\ (2n-3)i + n - 2j + 4, & \text{if } \frac{1}{2}(n + \lceil \frac{n}{2} \rceil) + 1 \leq j \leq n-1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k+3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 1(mod\ 12)$ and $n \geq 13$, is SVM. □

5.3.3 Linear $EL(kC_n)$ - Snake, $n \equiv 2$ or $8(mod\, 12)$ and $n \geq 14$

Theorem 5.3.3. *Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 2$ or $8(mod\, 12)$ and $n \geq 14$. Then EL(kC_n) is SVM.*

Proof. Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 2$ or $8(mod\, 12)$ and $n \geq 14$. Then $p + q = (2n - 3)k + 3$.

Define $f : E(EL(kC_n)) \rightarrow \{1, 2, 3, \cdots, (2n - 3)k + 3\}$ as follows;

Case 1a: For $i = 1$

$$f(e_{i,j}) = \begin{cases} 2j+1, & \text{if } 1 \leq j \leq \frac{n}{2}, \\ 2j, & \text{if } \frac{n}{2}+1 \leq j \leq \lfloor \frac{2n}{3} \rfloor, \\ 2j+1, & \text{if } \lfloor \frac{2n}{3} \rfloor + 1 \leq j \leq n-1, \\ 1, & \text{if } j = n. \end{cases}$$

Case 2a: For $2 \leq i \leq k - 1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \frac{n}{2} - 1, \\ (2n-3)i - n + 4, & \text{if } j = \frac{n}{2}, \\ (2n-3)i - 2n + 2j + 3, & \text{if } \frac{n}{2}+1 \leq j \leq \lfloor \frac{2n}{3} \rfloor, \\ (2n-3)i - 2n + 2j + 4, & \text{if } \lfloor \frac{2n}{3} \rfloor + 1 \leq j \leq n-1. \end{cases}$$

Case 3a: For $i = k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq n-1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} 2j, & \text{if } 1 \leq j \leq \frac{n}{2} - 1, \\ 2\lfloor \frac{2n}{3} \rfloor + 1, & \text{if } j = \frac{n}{2}, \\ 2n, & \text{if } j = \frac{n}{2} + 1, \\ 2j - 1, & \text{if } \frac{n}{2} + 2 \leq j \leq \lfloor \frac{2n}{3} \rfloor, \\ 2j, & \text{if } \lfloor \frac{2n}{3} \rfloor + 1 \leq j \leq n - 1, \\ n, & \text{if } j = n. \end{cases}$$

Case 2b: For $2 \leq i \leq k - 1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \frac{n}{2} - 1, \\ (2n-3)i - 2n + 2\lfloor \frac{2n}{3} \rfloor + 4, & \text{if } j = \frac{n}{2}, \\ (2n-3)i + 3, & \text{if } j = \frac{n}{2} + 1, \\ (2n-3)i - 2n + 2j + 2, & \text{if } \frac{n}{2} + 2 \leq j \leq \lfloor \frac{2n}{3} \rfloor, \\ (2n-3)i - 2n + 2j + 3, & \text{if } \lfloor \frac{2n}{3} \rfloor + 1 \leq j \leq n - 1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq n - 1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k + 3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 2$ or $8(mod\ 12)$ and $n \geq 14$, is SVM. □

5.3.4 Linear $EL(kC_n)$ - Snake, $n \equiv 3(mod\ 12)$ and $n \geq 15$

Theorem 5.3.4. *Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 3(mod\ 12)$ and $n \geq 15$. Then EL(kC_n) is SVM.*

Proof. Let EL(kC_n) be a linear edge linked cyclic snake, where $n \equiv 3(mod\ 12)$ and $n \geq 15$. Define $f : E(EL(kC_n)) \to \{1, 2, 3, \cdots, (2n-3)k + 3\}$ as follows;

Case 1a: For $i = 1$

$$f(e_{i,j}) = \begin{cases} n - 2j - 2, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 7, \\ 7, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 6, \\ 1, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 5, \\ 3j - 3\lfloor \frac{n}{2} \rfloor + 15, & \text{if } \lfloor \frac{n}{2} \rfloor - 4 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ n + 3, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ 3n - 2j, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n - 3, \\ 3n - 2j - 2, & \text{if } n - 2 \leq j \leq n. \end{cases}$$

Case 2a: For $2 \leq i \leq k - 1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i - n + 6, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 3, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \frac{n}{3} + \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i + n - 2j + 2, & \text{if } \frac{n}{3} + \lfloor \frac{n}{2} \rfloor \leq j \leq n - 1. \end{cases}$$

Case 3a: For $i = k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i + n - 2j + 2, & \text{if } \lfloor \frac{n}{2} \rfloor \leq j \leq \frac{1}{2}(n + \lfloor \frac{n}{2} \rfloor) - 1, \\ (2n-3)i + n - 2j + 1, & \text{if } \frac{1}{2}(n + \lfloor \frac{n}{2} \rfloor) \leq j \leq n - 1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} n - 2j + 1, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 7, \\ 10, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 6, \\ 4, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 5, \\ 3j - 3\lfloor \frac{n}{2} \rfloor + 14, & \text{if } \lfloor \frac{n}{2} \rfloor - 4 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ n + 5, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ 3n - 2j + 1, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n - 3, \\ n + 4, & \text{if } j = n - 2, \\ 3n - 2j - 1, & \text{if } n - 1 \leq j \leq n. \end{cases}$$

Case 2b: For $2 \leq i \leq k - 1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i - \frac{2n}{3} + 4, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 4, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \frac{n}{3} + \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 3, & \text{if } \frac{n}{3} + \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n - 2, \\ (2n-3)i - n + 4, & \text{if } j = n - 1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i - \lfloor \frac{n}{2} \rfloor + 2, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 3, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \frac{1}{2}(n + \lfloor \frac{n}{2} \rfloor), \\ (2n-3)i + n - 2j + 2, & \text{if } \frac{1}{2}(n + \lfloor \frac{n}{2} \rfloor) + 1 \leq j \leq n - 1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k + 3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 3(mod\ 12)$ and $n \geq 15$, is SVM. □

5.3.5 Linear $EL(kC_n)$ - Snake, $n \equiv 4(mod\,12)$ and $n \geq 16$

Theorem 5.3.5. *Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 4(mod\,12)$ and $n \geq 16$. Then EL(kC_n) is SVM.*

Proof. Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 4(mod\,12)$ and $n \geq 16$. Define $f : E(EL(kC_n)) \rightarrow \{1, 2, 3, \cdots, (2n-3)k+3\}$ as follows;

Case 1a: For $i = 1$

$$f(e_{i,j}) = \begin{cases} n-2j-1, & \text{if } 1 \leq j \leq \frac{n}{2}-6, \\ 2j-n+11, & \text{if } \frac{n}{2}-5 \leq j \leq \frac{n}{2}-3, \\ 2j-n+12, & \text{if } \frac{n}{2}-2 \leq j \leq \frac{n}{2}-1, \\ n+2, & \text{if } j = \frac{n}{2}, \\ 3n-2j+1, & \text{if } \frac{n}{2}+1 \leq j \leq n-2, \\ 3n-2j-1, & \text{if } n-1 \leq j \leq n. \end{cases}$$

Case 2a: For $2 \leq i \leq k-1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i-2n+2j+5, & \text{if } 1 \leq j \leq \frac{n}{2}-1, \\ (2n-3)i-n+7, & \text{if } j = \frac{n}{2}, \\ (2n-3)i+n-2j+4, & \text{if } \frac{n}{2}+1 \leq j \leq n-\frac{1}{2}(\lfloor\frac{n}{3}\rfloor+3), \\ (2n-3)i+n-2j+3, & \text{if } n-\frac{1}{2}(\lfloor\frac{n}{3}\rfloor+1) \leq j \leq n-3, \\ (2n-3)i+n-2j+2, & \text{if } n-2 \leq j \leq n-1. \end{cases}$$

Case 3a: For $i = k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i-2n+2j+5, & \text{if } 1 \leq j \leq \frac{n}{2}-1, \\ (2n-3)i+n-2j+3, & \text{if } \frac{n}{2} \leq j \leq \frac{3}{4}n-1, \\ (2n-3)i+n-2j+2, & \text{if } \frac{3}{4}n \leq j \leq n-1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} n - 2j, & \text{if } 1 \leq j \leq \frac{n}{2} - 6, \\ 6, & \text{if } j = \frac{n}{2} - 5, \\ 2j - n + 10, & \text{if } \frac{n}{2} - 4 \leq j \leq \frac{n}{2} - 3, \\ 2j - n + 11, & \text{if } \frac{n}{2} - 2 \leq j \leq \frac{n}{2} - 1, \\ n + 4, & \text{if } j = \frac{n}{2}, \\ 3n - 2j + 2, & \text{if } \frac{n}{2} + 1 \leq j \leq n - 2, \\ n + 3, & \text{if } j = n - 1, \\ n, & \text{if } j = n. \end{cases}$$

Case 2b: For $2 \leq i \leq k - 1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \frac{n}{2} - 1, \\ (2n-3)i - n + \lfloor \frac{n}{3} \rfloor + 5, & \text{if } j = \frac{n}{2}, \\ (2n-3)i + n - 2j + 5, & \text{if } \frac{n}{2} + 1 \leq j \leq n - \frac{1}{2}(\lfloor \frac{n}{3} \rfloor + 1), \\ (2n-3)i + n - 2j + 4, & \text{if } n - \frac{1}{2}(\lfloor \frac{n}{3} \rfloor - 1) \leq j \leq n - 2, \\ (2n-3)i - n + 5, & \text{if } j = n - 1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \frac{n}{2} - 1, \\ (2n-3)i - \frac{n}{2} + 3, & \text{if } j = \frac{n}{2}, \\ (2n-3)i + n - 2j + 4, & \text{if } \frac{n}{2} + 1 \leq j \leq \frac{3}{4}n, \\ (2n-3)i + n - 2j + 3, & \text{if } \frac{3}{4}n + 1 \leq j \leq n - 1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k + 3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 4(mod\ 12)$ and $n \geq 16$, is SVM. □

5.3.6 Linear $EL(kC_n)$ - Snake, $n \equiv 5(mod\, 12)$ and $n \geq 17$

Theorem 5.3.6. *Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 5(mod\, 12)$ and $n \geq 17$. Then $EL(kC_n)$ is SVM.*

Proof. Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 5(mod\, 12)$ and $n \geq 17$. Define $f : E(EL(kC_n)) \to \{1, 2, 3, \cdots, (2n-3)k+3\}$ as follows;

Case 1a: For $i = 1$

$$f(e_{i,j}) = \begin{cases} n-2j-2, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 7, \\ 7, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 6, \\ 1, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 5, \\ 3j - 3\lfloor \frac{n}{2} \rfloor + 15, & \text{if } \lfloor \frac{n}{2} \rfloor - 4 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ n+3, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ 3n-2j, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n-3, \\ 3n-2j-2, & \text{if } n-2 \leq j \leq n. \end{cases}$$

Case 2a: For $2 \leq i \leq k-1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 3, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq 10\lfloor \frac{n}{12} \rfloor + 3, \\ (2n-3)i + n - 2j + 2, & \text{if } 10\lfloor \frac{n}{12} \rfloor + 4 \leq j \leq n-2, \\ (2n-3)i - n + 3, & \text{if } j = n-1. \end{cases}$$

Case 3a: For $i = k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 4, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \frac{2n+5}{3}, \\ (2n-3)i + n - 2j + 3, & \text{if } \frac{2n+5}{3} + 1 \leq j \leq n-1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} n - 2j + 1, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 7, \\ 10, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 6, \\ 4, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 5, \\ 3j - 3\lfloor \frac{n}{2} \rfloor + 14, & \text{if } \lfloor \frac{n}{2} \rfloor - 4 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ n + 5, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ 3n - 2j + 1, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n - 3, \\ n + 4, & \text{if } j = n - 2, \\ 3n - 2j - 1, & \text{if } n - 1 \leq j \leq n. \end{cases}$$

Case 2b: For $2 \leq i \leq k - 1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i + n - 20\lfloor \frac{n}{12} \rfloor - 5, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 4, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq 10\lfloor \frac{n}{12} \rfloor + 4, \\ (2n-3)i + n - 2j + 3, & \text{if } 10\lfloor \frac{n}{12} \rfloor + 5 \leq j \leq n - 1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i - \lfloor \frac{n}{2} \rfloor + 3, & \text{if } j = \lfloor \frac{n}{2} \rfloor + 1, \\ (2n-3)i + n - 2j + 5, & \text{if } \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq \frac{2n+8}{3}, \\ (2n-3)i + n - 2j + 4, & \text{if } \frac{2n+11}{3} \leq j \leq n - 1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k + 3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 5(mod\ 12)$ and $n \geq 17$, is SVM. □

5.3.7 Linear $EL(kC_n)$ - Snake, $n \equiv 6(mod\,12)$ and $n \geq 18$

Theorem 5.3.7. *Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 6(mod\,12)$ and $n \geq 18$. Then EL(kC_n) is SVM.*

Proof. Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 6(mod\,12)$ and $n \geq 18$. Define $f : E(EL(kC_n)) \rightarrow \{1, 2, 3, \cdots, (2n-3)k+3\}$ as follows;

Case 1a: For $i = 1$

$$f(e_{i,j}) = \begin{cases} n-2j-1, & \text{if } 1 \leq j \leq \frac{n}{2}-6, \\ 2j-n+11, & \text{if } \frac{n}{2}-5 \leq j \leq \frac{n}{2}-3, \\ 2j-n+12, & \text{if } \frac{n}{2}-2 \leq j \leq \frac{n}{2}-1, \\ n+2, & \text{if } j = \frac{n}{2}, \\ 3n-2j+1, & \text{if } \frac{n}{2}+1 \leq j \leq n-2, \\ 3n-2j-1, & \text{if } n-1 \leq j \leq n. \end{cases}$$

Case 2a: For $2 \leq i \leq k-1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i-2n+2j+5, & \text{if } 1 \leq j \leq \frac{n}{2}-1, \\ (2n-3)i-n+7, & \text{if } j = \frac{n}{2}, \\ (2n-3)i+n-2j+4, & \text{if } \frac{n}{2}+1 \leq j \leq n-\frac{1}{2}(\lfloor\frac{n}{3}\rfloor+3), \\ (2n-3)i+n-2j+3, & \text{if } n-\frac{1}{2}(\lfloor\frac{n}{3}\rfloor+1) \leq j \leq n-3, \\ (2n-3)i+n-2j+2, & \text{if } n-2 \leq j \leq n-1. \end{cases}$$

Case 3a: For $i = k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i-2n+2j+5, & \text{if } 1 \leq j \leq \frac{n}{2}, \\ (2n-3)i+n-2j+5, & \text{if } \frac{n}{2}+1 \leq j \leq \lfloor\frac{3n}{4}\rfloor, \\ (2n-3)i+n-2j+4, & \text{if } \lfloor\frac{3n}{4}\rfloor+1 \leq j \leq n-1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} n - 2j, & \text{if } 1 \leq j \leq \frac{n}{2} - 6, \\ 6, & \text{if } j = \frac{n}{2} - 5, \\ 2j - n + 10, & \text{if } \frac{n}{2} - 4 \leq j \leq \frac{n}{2} - 3, \\ 2j - n + 11, & \text{if } \frac{n}{2} - 2 \leq j \leq \frac{n}{2} - 1, \\ n + 4, & \text{if } j = \frac{n}{2}, \\ 3n - 2j + 2, & \text{if } \frac{n}{2} + 1 \leq j \leq n - 2, \\ n + 3, & \text{if } j = n - 1, \\ n, & \text{if } j = n. \end{cases}$$

Case 2b: For $2 \leq i \leq k - 1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \frac{n}{2} - 1, \\ (2n-3)i - \frac{2n}{3} + 4, & \text{if } j = \frac{n}{2}, \\ (2n-3)i + n - 2j + 5, & \text{if } \frac{n}{2} + 1 \leq j \leq \frac{5n}{6}, \\ (2n-3)i + n - 2j + 4, & \text{if } \frac{5n}{6} + 1 \leq j \leq n - 1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \frac{n}{2}, \\ (2n-3)i - \frac{n}{2} + 4, & \text{if } j = \frac{n}{2} + 1, \\ (2n-3)i + n - 2j + 6, & \text{if } \frac{n}{2} + 2 \leq j \leq \lfloor \frac{3n}{4} \rfloor + 1, \\ (2n-3)i + n - 2j + 5, & \text{if } \lfloor \frac{3n}{4} \rfloor + 2 \leq j \leq n - 1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k + 3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 6(mod\ 12)$ and $n \geq 18$, is SVM. $\square$

5.3.8 Linear $EL(kC_n)$ - Snake, $n \equiv 7(mod\, 12)$ and $n \geq 19$

Theorem 5.3.8. *Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 7(mod\, 12)$ and $n \geq 19$. Then EL(kC_n) is SVM.*

Proof. Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 7(mod\, 12)$ and $n \geq 19$. Define $f : E(EL(kC_n)) \rightarrow \{1, 2, 3, \cdots, (2n-3)k+3\}$ as follows;

Case 1a: For $i = 1$

$$f(e_{i,j}) = \begin{cases} n-2j-2, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 7, \\ 7, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 6, \\ 1, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 5, \\ 3j - 3\lfloor \frac{n}{2} \rfloor + 15, & \text{if } \lfloor \frac{n}{2} \rfloor - 4 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ n+3, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ 3n-2j, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n-3, \\ 3n-2j-2, & \text{if } n-2 \leq j \leq n. \end{cases}$$

Case 2a: For $2 \leq i \leq k-1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i - n + 8, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 3, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i + n - 2j + 2, & \text{if } \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{2} \rfloor \leq j \leq n-4, \\ (2n-3)i + n - 2j + 1, & \text{if } n-3 \leq n-1. \end{cases}$$

Case 3a: For $i = k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i + n - 2j + 2, & \text{if } \lfloor \frac{n}{2} \rfloor \leq j \leq \frac{1}{2}(n + \lfloor \frac{n}{2} \rfloor) - 1, \\ (2n-3)i + n - 2j + 1, & \text{if } \frac{1}{2}(n + \lfloor \frac{n}{2} \rfloor) \leq j \leq n-1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} n - 2j + 1, & \text{if } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 7, \\ 10, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 6, \\ 4, & \text{if } j = \lfloor \frac{n}{2} \rfloor - 5, \\ 3j - 3\lfloor \frac{n}{2} \rfloor + 14, & \text{if } \lfloor \frac{n}{2} \rfloor - 4 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ n + 5, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ 3n - 2j + 1, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n - 3, \\ n + 4, & \text{if } j = n - 2, \\ 3n - 2j - 1, & \text{if } n - 1 \leq j \leq n. \end{cases}$$

Case 2b: For $2 \leq i \leq k - 1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i - 2\lfloor \frac{n}{3} \rfloor + 4, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 4, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 3, & \text{if } \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n - 3, \\ (2n-3)i + n - 2j + 2, & \text{if } n - 2 \leq j \leq n - 1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ (2n-3)i - \lfloor \frac{n}{2} \rfloor + 2, & \text{if } j = \lfloor \frac{n}{2} \rfloor, \\ (2n-3)i + n - 2j + 3, & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \frac{1}{2}(n + \lfloor \frac{n}{2} \rfloor), \\ (2n-3)i + n - 2j + 2, & \text{if } \frac{1}{2}(n + \lfloor \frac{n}{2} \rfloor) + 1 \leq j \leq n - 1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k + 3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 7(mod\ 12)$ and $n \geq 19$, is SVM. □

5.3.9 Linear $EL(kC_n)$ - Snake, $n \equiv 9(mod\,12)$ and $n \geq 21$

Theorem 5.3.9. *Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 9(mod\,12)$ and $n \geq 21$. Then $EL(kC_n)$ is SVM.*

Proof. Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 9(mod\,12)$ and $n \geq 21$. Define $f : E(EL(kC_n)) \to \{1, 2, 3, \cdots, (2n-3)k+3\}$ as follows;

Case 1a: For $i = 1$

$$f(e_{i,j}) = \begin{cases} n-2j, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 7, \\ 7, & \text{if } j = \lceil \frac{n}{2} \rceil - 6, \\ 1, & \text{if } j = \lceil \frac{n}{2} \rceil - 5, \\ 3j - 3\lceil \frac{n}{2} \rceil + 15, & \text{if } \lceil \frac{n}{2} \rceil - 4 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ n+3, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ 3n-2j+2, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq n-2, \\ 3n-2j, & \text{if } n-1 \leq j \leq n. \end{cases}$$

Case 2a: For $2 \leq i \leq k-1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 6, & \text{if } j = 1, \\ (2n-3)i - 2n + 2j + 5, & \text{if } 2 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i - n + 5, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ (2n-3)i + n - 2j + 5, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq \frac{n}{3} + \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i + n - 2j + 4, & \text{if } \frac{n}{3} + \lceil \frac{n}{2} \rceil \leq j \leq n-1. \end{cases}$$

Case 3a: For $i = k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i + n - 2j + 4, & \text{if } \lceil \frac{n}{2} \rceil \leq j \leq \frac{1}{2}(n + \lceil \frac{n}{2} \rceil) - 1, \\ (2n-3)i + n - 2j + 3, & \text{if } \frac{1}{2}(n + \lceil \frac{n}{2} \rceil) \leq j \leq n-1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} n-2j+1, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 7, \\ 10, & \text{if } j = \lceil \frac{n}{2} \rceil - 6, \\ 4, & \text{if } j = \lceil \frac{n}{2} \rceil - 5, \\ 3j - 3\lceil \frac{n}{2} \rceil + 14, & \text{if } \lceil \frac{n}{2} \rceil - 4 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ n+5, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ 3n-2j+3, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq n-2, \\ 4n-3j+1, & \text{if } n-1 \leq j \leq n. \end{cases}$$

Case 2b: For $2 \leq i \leq k-1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i - \frac{2n}{3} + 4, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ (2n-3)i + 3, & \text{if } j = \lceil \frac{n}{2} \rceil + 1 \text{ and } i+1 = k, \\ (2n-3)i + 4, & \text{if } j = \lceil \frac{n}{2} \rceil + 1 \text{ and } i+1 \neq k, \\ (2n-3)i + n - 2j + 6, & \text{if } \lceil \frac{n}{2} \rceil + 2 \leq j \leq \frac{n}{3} + \lceil \frac{n}{2} \rceil, \\ (2n-3)i + n - 2j + 5, & \text{if } \frac{n}{3} + \lceil \frac{n}{2} \rceil + 1 \leq j \leq n-1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i - \lceil \frac{n}{2} \rceil + 4, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ (2n-3)i + n - 2j + 5, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq \frac{1}{2}(n + \lceil \frac{n}{2} \rceil), \\ (2n-3)i + n - 2j + 4, & \text{if } \frac{1}{2}(n + \lceil \frac{n}{2} \rceil) + 1 \leq j \leq n-1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k+3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 9(mod\ 12)$ and $n \geq 21$, is SVM. □

5.3.10 Linear $EL(kC_n)$ - Snake, $n \equiv 10(mod\ 12)$ and $n \geq 22$

Theorem 5.3.10. *Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 10(mod\, 12)$ and $n \geq 22$. Then EL(kC_n) is SVM.*

Proof. Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 10(mod\ 12)$ and $n \geq 22$.

Define $f : E(EL(kC_n)) \to \{1, 2, 3, \cdots, (2n-3)k+3\}$ as follows;

Case 1a: For $i = 1$

$$f(e_{i,j}) = \begin{cases} n-2j-1, & \text{if } 1 \leq j \leq \frac{n}{2}-6, \\ 2j-n+11, & \text{if } \frac{n}{2}-5 \leq j \leq \frac{n}{2}-3, \\ 2j-n+12, & \text{if } \frac{n}{2}-2 \leq j \leq \frac{n}{2}-1, \\ n+2, & \text{if } j = \frac{n}{2}, \\ 3n-2j+1, & \text{if } \frac{n}{2}+1 \leq j \leq n-2, \\ 3n-2j-1, & \text{if } n-1 \leq j \leq n. \end{cases}$$

Case 2a: For $2 \leq i \leq k-1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i-2n+2j+5, & \text{if } 1 \leq j \leq \frac{n}{2}, \\ (2n-3)i+2n-2j+5, & \text{if } \frac{n}{2}+1 \leq j \leq 3\lfloor \frac{n}{4} + 1, \\ (2n-3)i+n-2j+4, & \text{if } 3\lfloor \frac{n}{4} \leq j \leq n-1. \end{cases}$$

Case 3a: For $i = k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i-2n+2j+5, & \text{if } 1 \leq j \leq \frac{n}{2}, \\ (2n-3)i+2n-2j+5, & \text{if } \frac{n}{2}+1 \leq j \leq 3\lfloor \frac{n}{4} \rfloor + 1, \\ (2n-3)i+n-2j+4, & \text{if } 3\lfloor \frac{n}{4} \rfloor \leq j \leq n-1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} n - 2j, & \text{if } 1 \le j \le \frac{n}{2} - 6, \\ 6, & \text{if } j = \frac{n}{2} - 5, \\ 2j - n + 10, & \text{if } \frac{n}{2} - 4 \le j \le \frac{n}{2} - 3, \\ 2j - n + 11, & \text{if } \frac{n}{2} - 2 \le j \le \frac{n}{2} - 1, \\ n + 4, & \text{if } j = \frac{n}{2}, \\ 3n - 2j + 2, & \text{if } \frac{n}{2} + 1 \le j \le n - 2, \\ n + 3, & \text{if } j = n - 1, \\ n, & \text{if } j = n. \end{cases}$$

Case 2b: For $2 \le i \le k - 1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \le j \le \frac{n}{2} - 1, \\ (2n-3)i - n + \lfloor \frac{n}{3} \rfloor + 5, & \text{if } j = \frac{n}{2}, \\ (2n-3)i + n - 2j + 5, & \text{if } \frac{n}{2} + 1 \le j \le n - \frac{1}{2}(\lfloor \frac{n}{3} \rfloor + 1), \\ (2n-3)i + n - 2j + 4, & \text{if } n - \frac{1}{2}(\lfloor \frac{n}{3} \rfloor - 1) \le j \le n - 2, \\ (2n-3)i - n + 5, & \text{if } j = n - 1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \le j \le \frac{n}{2}, \\ (2n-3)i - \frac{n}{2} + 4, & \text{if } j = \frac{n}{2} + 1, \\ (2n-3)i + n - 2j + 6, & \text{if } \frac{n}{2} + 2 \le j \le 3\lfloor \frac{n}{4} \rfloor + 2, \\ (2n-3)i + n - 2j + 5, & \text{if } 3\lfloor \frac{n}{4} \rfloor + 3 \le j \le n - 1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k + 3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 10(mod\ 12)$ and $n \ge 22$, is SVM.. □

5.3.11 Linear $EL(kC_n)$ - Snake, $n \equiv 11(mod\ 12)$ and $n \geq 23$

Theorem 5.3.11. *Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 11(mod\ 12)$ and $n \geq 23$. Then $EL(kC_n)$ is SVM.*

Proof. Let $EL(kC_n)$ be a linear edge linked cyclic snake, where $n \equiv 11(mod\ 12)$ and $n \geq 23$.

Define $f : E(EL(kC_n)) \rightarrow \{1, 2, 3, \cdots, (2n-3)k+3\}$ as follows;

Case 1a: For $i = 1$

$$f(e_{i,j}) = \begin{cases} n-2j, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 7, \\ 7, & \text{if } j = \lceil \frac{n}{2} \rceil - 6, \\ 1, & \text{if } j = \lceil \frac{n}{2} \rceil - 5, \\ 3j - 3\lceil \frac{n}{2} \rceil + 15, & \text{if } \lceil \frac{n}{2} \rceil - 4 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ n+3, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ 3n-2j+2, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq n-2, \\ 3n-2j, & \text{if } n-1 \leq j \leq n. \end{cases}$$

Case 2a: For $2 \leq i \leq k-1$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i - n + 8, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ (2n-3)i + n - 2j + 5, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq 10\lfloor \frac{n}{12} \rfloor + 8, \\ (2n-3)i + n - 2j + 4, & \text{if } 10\lfloor \frac{n}{12} \rfloor + 9 \leq j \leq n-3, \\ (2n-3)i + n - 2j + 3, & \text{if } n-2 \leq j \leq n-1. \end{cases}$$

Case 3a: For $i = k$

$$f(e_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 5, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 2, \\ (2n-3)i + n - 2j + 2, & \text{if } \lceil \frac{n}{2} \rceil - 1 \leq j \leq 9\lfloor \frac{n}{12} \rfloor + 7, \\ (2n-3)i + n - 2j + 1, & \text{if } 9\lfloor \frac{n}{12} \rfloor + 8 \leq j \leq n-1. \end{cases}$$

The induced vertex labels are as follows:

Case 1b: For $i = 1$

$$f^v(v_{i,j}) = \begin{cases} n - 2j + 1, & \text{if } 1 \leq j \leq \lceil \frac{n}{2} \rceil - 7, \\ 10, & \text{if } j = \lceil \frac{n}{2} \rceil - 6, \\ 4, & \text{if } j = \lceil \frac{n}{2} \rceil - 5, \\ 3j - 3\lceil \frac{n}{2} \rceil + 14, & \text{if } \lceil \frac{n}{2} \rceil - 4 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ n + 5, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ 3n - 2j + 3, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq n - 2, \\ 4n - 3j + 1, & \text{if } n - 1 \leq j \leq n. \end{cases}$$

Case 2b: For $2 \leq i \leq k - 1$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i - 20\lfloor \frac{n}{12} \rfloor + n - 13, & \text{if } j = \lceil \frac{n}{2} \rceil, \\ (2n-3)i + n - 2j + 6, & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq j \leq 10\lfloor \frac{n}{12} \rfloor + 9, \\ (2n-3)i + n - 2j + 5, & \text{if } 10\lfloor \frac{n}{12} \rfloor + 10 \leq j \leq n - 2, \\ (2n-3)i - n + 6, & \text{if } j = n - 1. \end{cases}$$

Case 3b: For $i = k$

$$f^v(v_{i,j}) = \begin{cases} (2n-3)i - 2n + 2j + 4, & \text{if } 2 \leq j \leq \lceil \frac{n}{2} \rceil - 2, \\ (2n-3)i - \lceil \frac{n}{2} \rceil + 3, & \text{if } j = \lceil \frac{n}{2} \rceil - 1, \\ (2n-3)i + n - 2j + 3, & \text{if } \lceil \frac{n}{2} \rceil \leq j \leq 9\lfloor \frac{n}{12} \rfloor + 8, \\ (2n-3)i + n - 2j + 2, & \text{if } 9\lfloor \frac{n}{12} \rfloor + 9 \leq j \leq n - 1. \end{cases}$$

It can be easily verified that f is a Super Vertex Mean labeling as it is an injective mapping and the set of edge labels and induced vertex labels is $\{1, 2, 3, ..., (2n-3)k + 3\}$. Therefore linear $EL(kC_n)$ - snake, $n \equiv 11 (mod\ 12)$ and $n \geq 23$, is SVM. □

5.4 Conclusion

We have so far successfully proved that all the edge linked linear cyclic snakes are Super Vertex Mean graphs. A researcher is further encouraged to attempt to prove the SVM - behaviour of non-linear edge linked cyclic snakes. In the case of non-linear $EL(kC_n)$ - snake, the value of s_i' for each i may not be equal to $\lfloor \frac{n}{2} \rfloor - 1$ and $\lceil \frac{n}{2} \rceil - 1$.

Chapter 6

SVM Graphs of Order ≤ 7

For a researcher a natural quest arises to examine all the graphs up to a certain order as to find out how many of them fall into the category of SVM - graphs. Therefore, in this chapter we investigate the SVM - behaviour of all graphs up to order 5 and all regular graphs up to order 7.

6.1 Preliminary Observations

6.1.1 Necessary Condition

If $d(v) = 0$ for any vertex v of G then it is called an isolated vertex and if $d(v) = 1$ then it is called a pendant vertex. From the definition of Super Vertex Mean labeling it is clear that a graph containing a vertex v whose $d(v) \leq 1$ cannot be a Super Vertex Mean graph. Therefore, necessarily $deg(v) \geq 2$ for all vertices v of a SVM graph G. It is obvious that no tree is a SVM graph. In this chapter, we discuss only those graphs G with $d(v) \geq 2$ for all vertices v of G.

6.1.2 Regularity of Graphs

If $d(v)$ = r, for every vertex v of a graph G, then G is called a r-regular graph. From the above observation, we know that no zero regular or 1-regular graph is an SVM graph. A

(p, q) - graph G can be r-regular graph if and only if $p \times r$ is even. It is derived from the fact that 'Odd order graphs cannot be odd-regular graphs [Theorem 2.6] of [8].' Consequently the number of edges of a r-regular graph is $(\frac{p \times r}{2})$, i.e., $q = (\frac{p \times r}{2})$.

6.1.3 Cycles are SVM

All the cycles $C_n, n \geq 3$ are 2-regualar graphs as the degree of each vertex is 2. In our previous chapters we have proved that all cycles, C_n for any $n \geq 3$, except C_4 are SVM - graphs.

6.2 List of Regular Graphs of Order ≤ 7

6.2.1 Of order 3

When order of a graph G is 3, there is just one 2-regular graph. This is a cycle of length 3, known as C_3 or K_3. We have already proved that it is an SVM - graph.

6.2.2 Of order 4

There are two regular graphs of order 4, of which C_4 is 2-regular and K_4 is 3-regular. We have proved that C_4 cannot be a SVM - graph.

6.2.3 Of order 5

We have a 4-regular graph K_5 with 10 edges and a 2-regular graph C_5 with 5 edges of order 5, of which C_5 have been proved to be a SVM - graph.

6.2.4 Of order 6

There are a total of 6 regular graphs of order 6. They are C_6, the disjoint union of two C_3's, both of which are 2-regular, two non-isomorphic 3-regular graphs with 9 edges each, a 4-regular graph with 12 edges and the 5-regular graph K_6 with 15 edges.

6.2.5 Of order 7

The number of regular graphs of order 7 is 5. They are C_7, disjoint union of C_3 and C_4, both of which are 2-regulars, two non-isomorphic 4-regular graphs with 14 edges and the complete graph K_7, which is 6-regular.

Now we proceed to prove that all these regular graphs are Super Vertex Mean graphs, excepting C_4. Before that we discuss the behaviour of disjoint union of graphs.

6.3 Disjoint Union of Graphs

The disjoint union of m copies of a graph G is denoted by mG. The union of two graphs G_1 and G_2 is a graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Theorem 6.3.1. *If G is an SVM - graph, so is mG and if G_1 and G_2 are SVM graphs, so is $G_1 \cup G_2$. The converse is not true.*

Proof. For the first part of the theorem, it is enough to prove that if G_1 and G_2 are two SVM - graphs, then $G_1 \cup G_2$ is also SVM.
Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two SVM graphs with Super Vertex Mean labelings f and g respectively on them. Let,

$$E(G_1) = \{e_i : 1 \leq i \leq q_1\}, V(G_1) = \{u_i : 1 \leq i \leq p_1\},$$

$$E(G_2) = \{e'_i : 1 \leq i \leq q_2\}, V(G_2) = \{u'_i : 1 \leq i \leq p_2\}.$$

Define $h : E(G_1 \cup G_2) \to \{1, 2, 3, \cdots, p_1 + q_1 + p_2 + q_2\}$ by

$$h(e_i) = f(e_i), for\ 1 \leq i \leq q_1,\ h(e'_i) = p_1 + q_1 + g(e'_i), for\ 1 \leq i \leq q_2$$

Now we show that h is an injection. Let,

$$h(e_i) = h(e_j)\ \Rightarrow f(e_i) = f(e_j)$$

Since, f is an injection, we have, $e_i = e_j$.

Let, $h(e'_i) = h(e'_j) \Rightarrow p_1 + q_1 + g(e'_i) = p_1 + q_1 + g(e'_j)$

$$\Rightarrow g(e'_i) = g(e'_j)$$

Since, g is an injection, we have,

$$e'_i = e'_j.$$

Therefore h is also and injection.

Suppose

$$h(e'_i) = h^v(u'_j)$$

$$\Rightarrow p_1 + q_1 + g(e'_i) = p_1 + q_1 + g^v(u'_j)$$

$$\Rightarrow g(e'_i) = g^v(u'_j)$$

which is a contradiction as g is Super Vertex Mean labeling.

So h is a SVM labeling.

To prove the second part of the theorem, we prove that although C_4 is not a SVM - graph, $2C_4$ and $C_3 \cup C_4$ are SVM - graphs.

Also we prove the general case that $C_3 \cup C_m$ is SVM for all $m \geq 3$.

We know that C_m is SVM graph for all $m \geq 3$ and $m \neq 4$. Therefore it is enough to prove that $C_3 \cup C_4$ and $2C_4$ are SVM - graphs.

Case 1: $C_3 \cup C_4$ is a SVM - graph.

Let,

$$E(C_3) = \{e_1, e_2, e_3\}$$

and

$$E(C_4) = \{e'_1, e'_2, e'_3, e'_4\}$$

Define $f : E(C_3 \cup C_4) \rightarrow \{1, 2, 3, \cdots, 13, 14\}$ by

$$f(e_1) = 1, f(e_2) = 3, f(e_3) = 7$$

$$f(e'_1) = 6, f(e'_2) = 10, f(e'_3) = 14, f(e'_4) = 11$$

It is clear that f is a Super Vertex Mean labeling of $C_3 \cup C_4$. Therefore $C_3 \cup C_4$ is SVM - graph, though C_4 is not. □

Example 6.3.2. *Super vertex mean labeling of* $C_3 \cup C_4$ *is shown in Figure* 6.1.

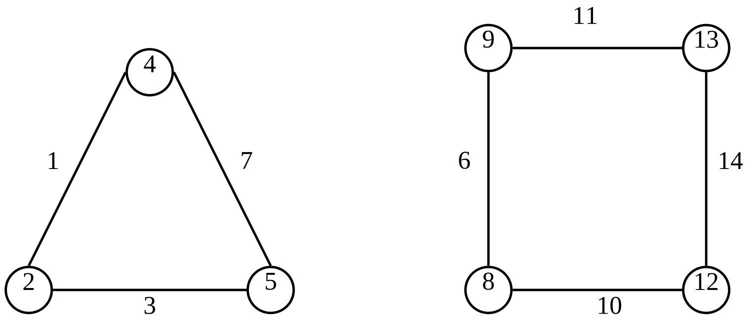


Figure 6.1: Super vertex mean labeling of $C_3 \cup C_4$

Case 1a. General Case: $C_3 \cup C_m$ is SVM for all $m \geq 3$ including $m = 4$.
All cycles, except C_4, are SVM - graphs and so their union, but then $C_3 \cup C_4$ is a SVM - graph. So, it is a clear fact that $C_3 \cup C_m$ is SVM for all $m \geq 3$. But we want to prove it in an alternate way without deriving from the above theorem and the fact that C_3 and C_m are SVM - graphs for all $m \neq 4$.

Proof. There is nothing to prove in the case of odd m as all odd cycles are SVM - graphs and their union is also SVM. Without loss of generality, we assume that m is even and $m \geq 4$.
Let $m = 2n$ for some $n \geq 2$.
Let

$$E(C_3) = \{e_1, e_2, e_3\}$$

and

$$E(C_m) = \{e'_1, e'_2, \cdots, e'_m = e'_{2n}\}$$

Define $f : E(C_3 \cup C_m) \to \{1, 2, 3, \cdots, 2m + 6 = 4n + 6\}$ by

$$f(e_1) = 1, f(e_2) = 3, f(e_3) = 7$$

$$f(e'_i) = \begin{cases} 6 & \text{if } i = 1 \\ 4i + 2 & \text{if } 2 \leq i \leq n+1 \\ 8n - 4i + 11 & \text{if } n+2 \leq i \leq 2n = m \end{cases}$$

Thus f is a super vertex mean labeling of $C_3 \cup C_m$ for all even $m \geq 4$, and it is an SVM graph. □

Case 2: $2C_4$ is a SVM - graph.

Proof. Let C_4 and C'_4 be two cycles of length 4.
Let

$$E(C_4) = \{e_1, e_2, e_3, e_4\}$$

and

$$E(C'_4) = \{e'_1, e'_2, e'_3, e'_4\}$$

Define $f : E(C_4 \cup C'_4) \to \{1, 2, 3, \cdots, 15, 16\}$
by

$$f(e_1) = 1, f(e_2) = 3, f(e_3) = 5, f(e_4) = 10$$

$$f(e'_1) = 7, f(e'_2) = 11, f(e'_3) = 14, f(e'_4) = 16$$

Then f is a Super Vertex labeling of $2C_4$, and $2C_4$ is a SVM - graph. □

Example 6.3.3. *Super vertex mean labeling of* $2C_4$ *is shown in Figure 6.2.*

Corollary 6.3.4. mC_4 *is a SVM - graph for all even* $m \geq 2$*.*

Proof. By the above theorem, we have proved that $2C_4$ and union of any two SVM - graphs is a SVM - graph.

Any even m is a multiple of 2, and therefore mC_4 is a union of $\frac{m}{2}$ times of $2C_4$. Or, mC_4, for $m \geq 4$, $m \equiv 0 \ (mod\ 2)$ is equal to $(m-2)C_4 \cup 2C_4$, where both of which are SVM - graphs. Thus the corollary. □

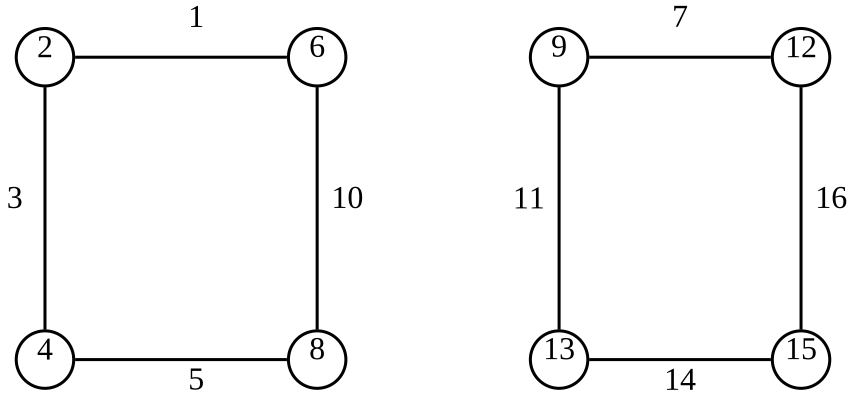


Figure 6.2: Super vertex mean labeling of $2C_4$.

Corollary 6.3.5. *Disjoint union of any number of cycles of any length, except C_4 is a SVM - graph.*

Proof. Since all the cycles except C_4 are SVM - graphs, by the above theorem, their unions are SVM - graphs. Thus the corollary. □

Corollary 6.3.6. *When the disjoint union of any number of cycles of any length contains C_4, it is a SVM - graph when,*

1. *There are even number of C_4 in the union, or*

2. *There exists at least one C_3 in the union.*

Proof. **1.** If there are even number of C_4 in the union, by the above corollary 1, union of these is SVM graph. All other cycles are SVM graphs. Therefore the union of both is a SVM graph by above theorem.

2. If there exists at least one C_3 in the union of cycles, then the union of this C_3 and any one C_4, if C_4 has an odd occurrence, is a SVM - graph. Otherwise, C_4 occurs in even number of times, and their union is proved to be a SVM - graph. □

6.4 Regular Graphs as SVM - Graphs

Theorem 6.4.1. *Regular graphs of order ≤ 7 and Petersen graph are SVM -graphs, C_4 being the only exception.*

6.4.1 Petersen graph

Given in Figure 6.3 is an SVM labeling of $3-$ regular graph of order 10, known as Petersen Graph.

Since $f(U) \cup f^v(V) = \{1, 2, 3, \cdots, 24, 25\}$, it is a SVM - labeling. While labeling Petersen

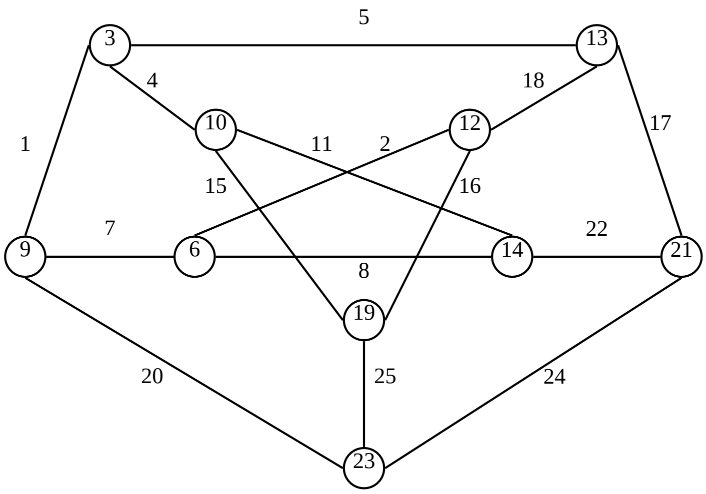


Figure 6.3: SVM labeling of Petersen Graph.

graph, it is interesting to observe that the sum of all vertex labels is $\frac{2}{3}$ of the sum of all vertex labels.

i.e.,

$$\sum_{v \in V(G)} f^v(v) = \left(\frac{\sum_{e \in E(G)} f(e) \times 2}{3} \right)$$

It happens because when we calculate the induced vertex label which is rounded up average of the labels of $3-$ edges that are incident on that particular vertex, we consider the edges twice.

Being a SVM labeling, sum of all these labels is,

$$\begin{aligned}\left(\frac{(p+q)(p+q+1)}{2}\right) &= \sum_{v\in V(G)} f^v(v) + \sum_{e\in E(G)} f(e)\\ &= \left(\frac{\sum_{e\in E(G)} f(e)\times 2}{3}\right) + \sum_{e\in E(G)} f(e)\\ &= \left(\frac{\sum_{e\in E(G)} f(e)\times 5}{3}\right)\end{aligned}$$

Here for Petersen graph, the total is 325, and sum of all edge labels is 195 and that of all vertex labels is 130, perfectly in agreement with the above observation.

This need not be a necessary phenomenon for all types of SVM - labeling of regular graphs. But this happens true for most of the regular graphs which we have examined. This fact is used as a hint for labeling the following graphs of order up to 7.

6.5 Regular graphs of order 3.

The only $2-$ regular graph of order 3 is the cycle C_3. We know that C_3 is a SVM - graph.

6.6 Regular graphs of order 4.

Regular graphs of order 4 are C_4, which is 2 - regular and K_4, that is 3 - regular. We know that C_4 is not a SVM - graph. We prove that 3 - regular graph of order 4, i.e., K_4 is a SVM - graph.

By above observation,

$$\begin{aligned}\left(\frac{(p+q)(p+q+1)}{2}\right) &= \sum_{v\in V(G)} f^v(v) + \sum_{e\in E(G)} f(e)\\ &= \left(\frac{\sum_{e\in E(G)} f(e)\times 2}{3}\right) + \sum_{e\in E(G)} f(e)\\ &= \left(\frac{\sum_{e\in E(G)} f(e)\times 5}{3}\right)\end{aligned}$$

In this case of K_4, we have,

$$\sum_{e\in E(G)} f(e) = \frac{3}{5} \times \left(\frac{(p+q)(p+q+1)}{2}\right)$$

ie.,

$$\sum_{e\in E(G)} f(e) = \frac{3}{5} \times \left(\frac{10 \times 11}{2}\right)$$
$$= 33$$

and,

$$\sum_{v\in V(G)} f^v(v) = \frac{2}{5} \times \left(\frac{10 \times 11}{2}\right)$$
$$= 22$$

So we can select the set $\{4, 5, 6, 7\}$, as the vertex label set, the sum of whose elements is equal to 22. Consequently the edge label set is $\{1, 2, 3, 8, 9, 10\}$, sum of whose elements is 33. We have partitioned the positive integers up to $p + q$, (here it is 10) in the above manner by the following logic. These numbers, 1 to 10, have to be distributed into two mutually disjoint sets in such a way that except any 4 numbers that are reserved as induced vertex labels, have to be clubbed in 4 sets of 3 elements (K_4 is a 3 - regular graph) and have to appear exactly twice without two numbers of one set coming together in some other set. it is because two vertices are connected by a single edge. And in a complete graph like K_4, each vertex is connected by an edge to every other vertex of the graph.

It is impossible to include the numbers $1, 2, 9$ and 10 in vertex label set. While calculating the average we cannot obtain one of these numbers as the rounded up average of any 3 numbers up to 10, without including the same number. Using the same number both as vertex label and edge label is ruled out by the definition of SVM - labeling.
Therefore,

$$\{1, 2, 9, 10\} \subseteq f(E)$$

The sum of these numbers is 22. If we take two more numbers in the edge label set (K_4 has 6 edges), so that the sum equal to 33, we are done. By careful way of inspection, we

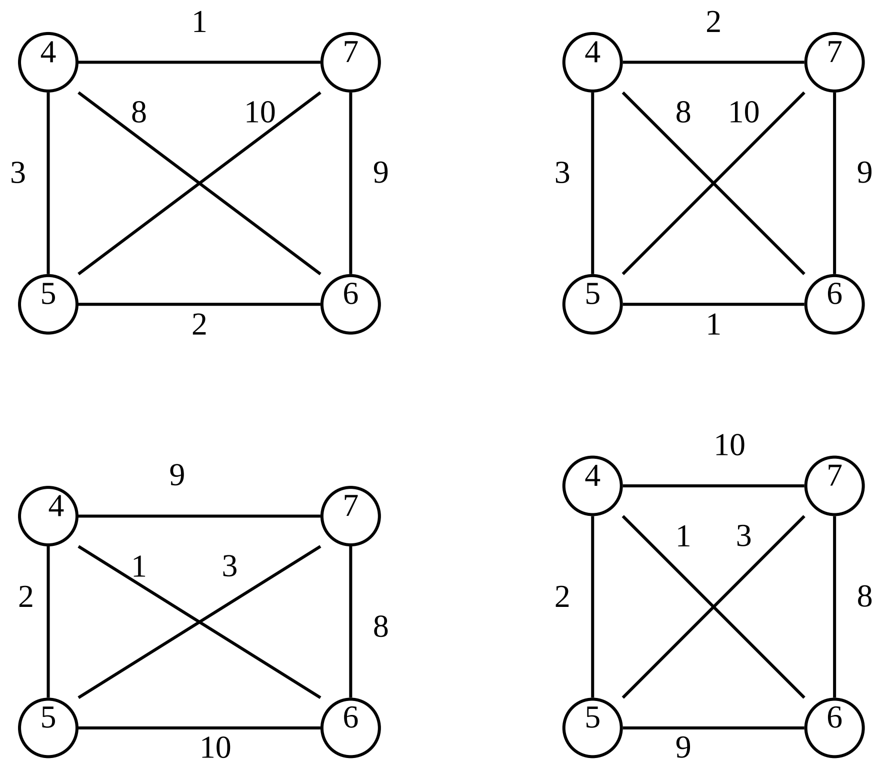


Figure 6.4: K_4 is SVM - labeled in 4 different ways.

have found that the only possibility is to include the numbers 3 and 8 in to the above set. So,

$$f(E) = \{1, 2, 3, 8, 9, 10\}$$

and,

$$f^v(V) = \{4, 5, 6, 7\}$$

Using these sets, we can label K_4 in 4 different ways as shown below in Figure 6.4.

6.7 Regular graphs of order 5

The r - regular graphs, $3 \leq r \leq p-1$ of order $p = 5$ are C_5, which is a $2-$ regular graph, and K_5 which is $4-$ regular graph. The graph being of an odd order, there cannot be any odd regular graphs. In our previous chapters we have proved that C_5, a cycle of length 5 is SVM graph. Now we proceed to prove that K_5, the complete graph of order 5 is SVM graph.

We observe that any complete graph K_n for some $n \geq 3$ is a $n-1$ regular graph. Therefore it has $\frac{n\times(n-1)}{2}$ edges.

Therefore,

$$p+q = n + \frac{n \times (n-1)}{2}$$

Equivalently, for any $r-$ regular graph,

$$p+q = n + \frac{n \times r}{2}$$

$$= \frac{n \times (r+2)}{2}$$

When $r = n-1$, we get

$$p+q = \frac{n \times (n+1)}{2}$$

For K_5,

$$p+q = \frac{5 \times 6}{2} = 15$$

As in the case of K_4, here

$$\sum_{v\in V(G)} f^v(v) = \frac{\sum_{e\in E(G)} f(e) \times 2}{4}$$

may be true. It is because every edge is counted twice while finding the induced vertex label which is the rounded up average of labels of 4 edges incident on that particular vertex.

Therefore,

$$\frac{(p+q)\times(p+q+1)}{2} = \sum_{v\in V(G)} f^v(v) + \sum_{e\in E(G)} f(e)$$

$$= \sum_{e\in E(G)} f(e) + \frac{1}{2} \times \sum_{e\in E(G)} f(e)$$

$$= \frac{3}{2} \times \sum_{e\in E(G)} f(e)$$

$$\Rightarrow \sum_{e\in E(G)} f(e) = \frac{2}{3} \times \frac{(p+q)\times(p+q+1)}{2}$$

$$Now, \ \frac{(p+q)\times(p+q+1)}{2} = \frac{15 \times 16}{2} = 120$$

$$\sum_{e \in E(G)} f(e) = 80$$

$$\sum_{v \in V(G)} f^v(v) = 40$$

From the set $\{1, 2, 3, \cdots, 14, 15\}$, the subset $\{1, 2, 14, 15\}$ has to be a subset of $f(E)$ in SVM labeling. If 3 becomes a vertex label, then 5 and 6 cannot become vertex labels because when 3 and 6 or 3 and 5 become vertex labels, then among $10, 11, 12$ and 13, only three numbers could be chosen as induced vertex labels.

For example,

$$3 = Round\left(\frac{1+2+3+4}{4}\right)$$

$$6 = Round\left(\frac{1+7+8+9}{4}\right)$$

If we select 10 and 13 as the next two vertex labels then only 11 can be the fifth one, i.e.,

$$13 = Round\left(\frac{9+12+14+15}{4}\right)$$

$$13 = Round\left(\frac{11+12+14+15}{4}\right)$$

Then

$$13 = Round\left(\frac{9+12+14+15}{4}\right)$$

The remaining numbers that could be used to get rounded up average of 10 and 11 are $2, 4, 5, 7, 8, 12, 14$ and 15, and they can be classified into 3 sets which appeared elsewhere. So we cannot have any option to have rounded up average of 10 and 11 without repeating any numbers which already appeared in pair.

If we select 11 and 13 as vertex labels where,

$$13 = Round\left(\frac{9+12+14+15}{4}\right)$$

or,

$$13 = Round\left(\frac{10+12+14+15}{4}\right)$$

then 12 cannot be the fifth vertex label. 10 is already ruled out to be the vertex label with 13 as another vertex label.

Therefore, 13 has to be an edge label and $10, 11$ and 12 can be vertex labels along with 3. Here too, 12 has only three options left,

$$12 = Round\left(\frac{4+13+14+15}{4}\right)$$

$$12 = Round\left(\frac{5\ or\ 6+13+14+15}{4}\right)$$

$$12 = Round\left(\frac{7+13+14+15}{4}\right)$$

This implies 10 and 11 are obtained as averages by making use of any one of the numbers among $13, 14$ and 15.

For example, 11 cannot be made a vertex label without repeating any one of the above numbers.

Therefore when 3 becomes a vertex label, the only next vertex label can be 7 or any number greater than 7. By continuing our inspection in a similar way we get the possible sets which can be vertex label set as follows;

1. $\{3, 7, 8, 10, 12\}$
2. $\{3, 7, 9, 10, 11\}$
3. $\{4, 6, 7, 10, 13\}$
4. $\{4, 6, 7, 11, 12\}$
5. $\{4, 6, 8, 9, 13\}$
6. $\{4, 6, 8, 10, 12\}$
7. $\{4, 6, 9, 10, 11\}$
8. $\{6, 7, 8, 9, 10\}$
9. $\{6, 7, 8, 9, 11\}$

It is interesting to note that except the 9^{th} set, all the others follow the rule,

$$\sum_{v\in V(G)} f^v(v) = \frac{1}{2} \times \sum_{e\in E(G)} f(e) = 40$$

Therefore for r - regular graphs, the condition

$$\sum_{v\in V(G)} f^v(v) = \frac{2}{r} \times \sum_{e\in E(G)} f(e)$$

is not a necessary condition, but only a hint to SVM labeling.

Example 6.7.1. *Given below in Figure* 6.5 *are the pictorial representations of nine different SVM - labelings of* K_5.

6.8 Regular graphs of order 6

Regular graphs having no isolated or pendant vertex of order 6 are the cycle, C_6 and $2C_3$, which are 2 - regulars, $K_{3,3}$ and another graph with 9 edges, both of them are 3 - regulars, the octahedral graph with 12 edges, which is 4- regular and the complete graph K_6. In total there are 6 non-isomorphic r-regular graphs of order 6, where $2 \leq r \leq 5$.
We have already proved that C_6 and $2C_3$ are SVM graphs. We show now that $K_{3,3}$ is a SVM graph.

6.8.1 $K_{3,3}$

For $K_{3,3}$, $p = 6$ and $q = 9$.
Therefore,

$$f(E) \cup f^v(V) = \{1, 2, 3, \cdots, 14, 15\}$$

While inspecting the possibility of SVM labeling of $K_{3,3}$ we have to keep the following in mind:
1. Partition the above set into two sets, keeping the hint for labeling r- regular graphs, i.e., $\sum_{v \in V(G)} f^v(v) = \frac{2}{r} \times \sum_{e \in E(G)} f(e)$
2. Clearly $f^v(V)$ contains 6 elements and $f(E)$ has 9 elements.
3. Now the set $f(E)$ is distributed into six sets of 3 elements each in such a way that,
3. a) The rounded up average of each set is one of the numbers in the set $f^v(V)$. These numbers are not repeated.
3. b)These six sets form two partitions, each partition having 3 sets and no number in one set of one partition is repeated in another set of the same partition.

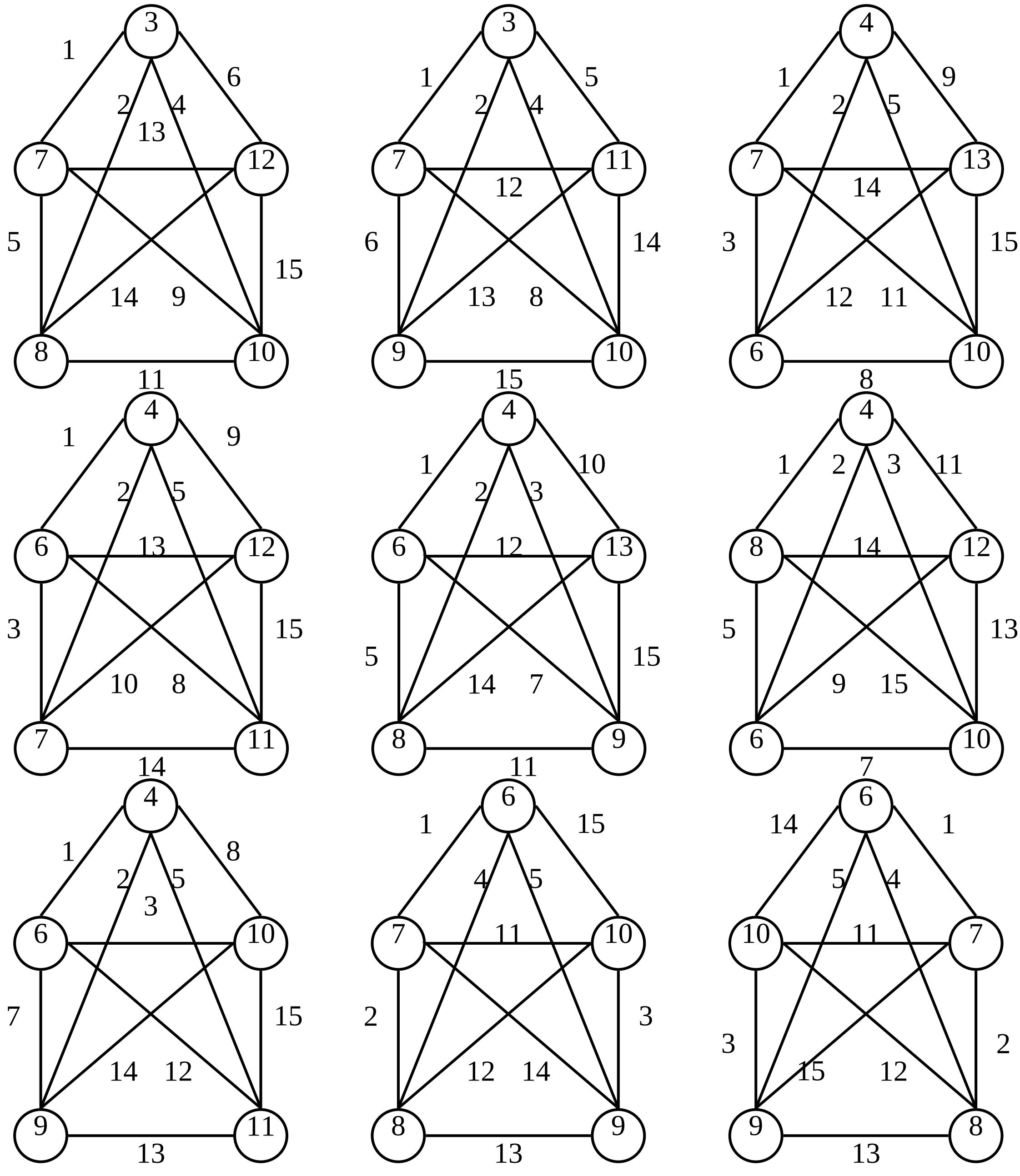


Figure 6.5: 9 different SVM - labelings of K_5

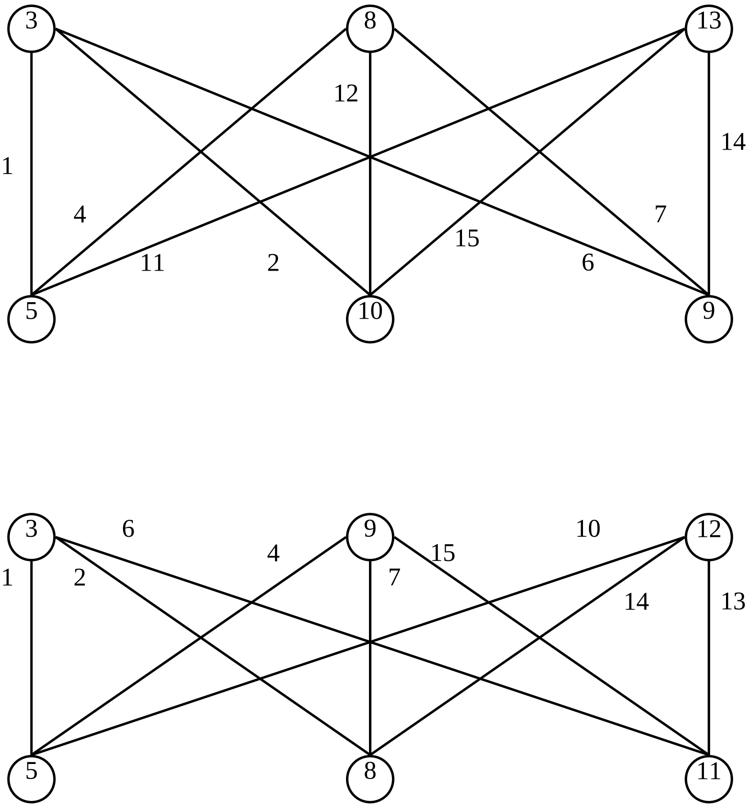


Figure 6.6: Pictorial representation of SVM - labelings of $K_{3,3}$

3. c) All the three numbers in one set of one partition are distributed equally in each set of the second partition.

Following above directions we form six subsets of $f(E)$ as given below;

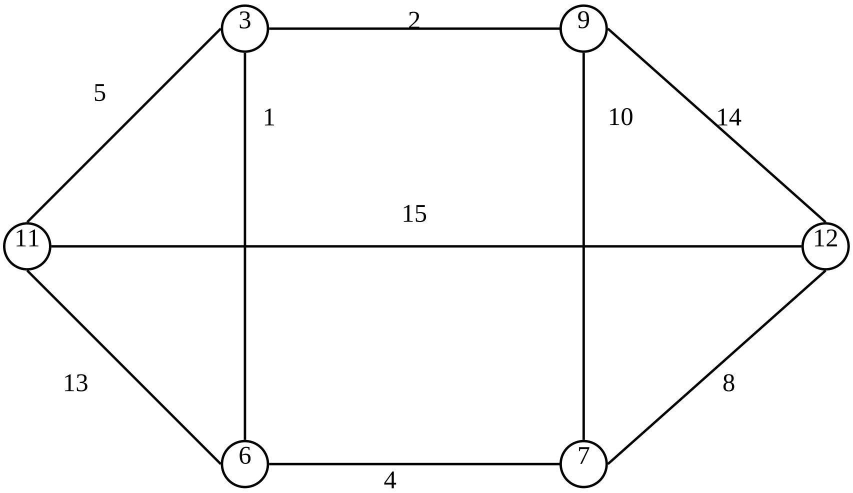
$\{1, 2, 6\}, \{4, 7, 12\}, \{11, 14, 15\}$ and $\{1, 4, 11\}, \{6, 7, 14\}, \{2, 12, 15\}$ whose rounded up averages are $3, 8, 13, 5, 9$ and 10 respectively.

Note that unions of the first three sets and the last three sets are having the same elements, the only difference being that two elements of any set do not appear together in any other set. The first three sets and the last three sets in themselves form two different partitions of the set $f(E)$.

Example 6.8.1. *The above labelings are shown pictorially in Figure* 6.6*. One more SVM - labeling of* $K_{3,3}$ *is given in the same Figure.*

6.8.2 Another 3-regular graph of order 6

There is another 3 - regular graph that is non-isomorphic to $K_{3,3}$ with 9 edges and 6 vertices.Therefore we cannot use the same method that we used in the case of the previous graph. The SVM labelings of this graph is shown below in Figure 6.7.



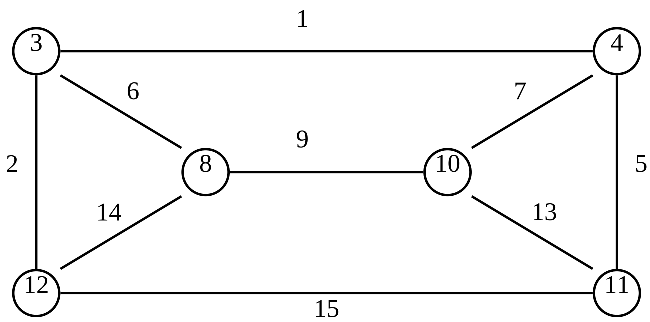


Figure 6.7: SVM - labelings of another 3-regular graph of order 6

6.8.3 4-regular graph of order 6 – Octahedral graph

There is yet another graph of order 6 and having 12 edges, which is a 4 -regular graph. This graph is known as Octahedral graph. For this graph,

$$p + q = 18$$

$$\frac{(p+q)\times(p+q+1)}{2} = 171$$

$$\sum_{v\in V(G)} f^v(v) = \frac{2}{r} \times \sum_{e\in E(G)} f(e)$$

$$= \frac{2}{4} \times \sum_{e\in E(G)} f(e)$$

$$171 = \frac{3}{2} \times \sum_{e\in E(G)} f(e)$$

$$\sum_{v\in V(G)} f^v(v) = \frac{171}{3} = 57$$

Using this hint we can partition the numbers up to 18 into two sets as given below;

$$f(E) = \{1, 2, 4, 5, 6, 7, 12, 13, 14, 15, 18\}$$

$$f^v(V) = \{3, 8, 9, 10, 11, 16\}$$

where $f(E)$ contains q elements and $f^v(V)$ has p elements. The elements of $f(E)$ are repeated exactly once to find the rounded up average of four numbers of $f(E)$, in order to obtain the elements in $f^v(V)$. Care should be taken so as not to place two numbers together while finding a second rounded up average. Thus we find that this 4- regular graph of order 6 too is a SVM - graph with the SVM - labeling as shown in Figure 6.8.

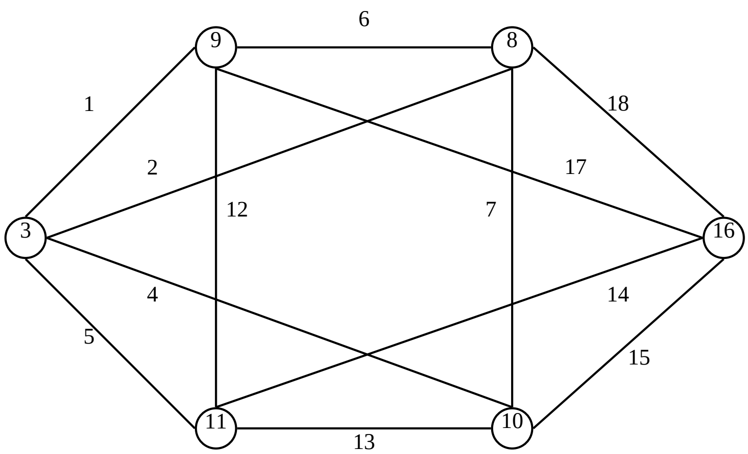


Figure 6.8: SVM - labeling of a 4-regular graph of order 6 – Octahedral Graph

6.8.4 The Complete graph K_6

Now we have the task of labeling K_6, the complete graph of order 6. Being a 5- regular graph the hint that we could use is that,

$$\sum_{v\in V(G)} f^v(v) = \frac{2}{5} \times \sum_{e\in E(G)} f(e)$$

The total sum of all the numbers up to $p+q$, i.e., up to 21 is 231, where $p = 6$ and $q = 15$. By the definition of SVM - labeling, we have,

$$231 = \sum_{v\in V(G)} f^v(v) + \sum_{e\in E(G)} f(e)$$

This implies that,

$$\frac{7}{5} \times \sum_{e\in E(G)} f(e) = 231$$

$$\sum_{e\in E(G)} f(e) = \frac{231 \times 5}{7} = 165$$

and,

$$\sum_{v\in V(G)} f^v(v) = \frac{231 \times 2}{7} = 66$$

So we partition the numbers up to 21 into two sets,

$$f(E) = \{1, 2, 3, 5, 6, 7, 8, 9, 14, 15, 17, 18, 19, 20, 21\}$$

$$f^v(V) = \{4, 10, 11, 12, 13, 16\}$$

having q and p elements respectively and the respective sum of its members being 165 and 66.

The other aspects are also kept in mind as in the previous cases of labeling regular and complete graphs.

In a complete graph's SVM - labeling, the $(n - 1)$ elements of $f(E)$ that are taken to calculate the rounded up average (in order to get one of the elements of $f^v(V)$) are used a total of $(n - 1)$ instances. But they are used one at a time, and without repeating. Whereas in a r-regular graph's labeling only r - elements are used only in any r - instances, one at a time and without repeating. Thus we obtain a SVM - labeling of K_6 as shown in Figure 6.9.

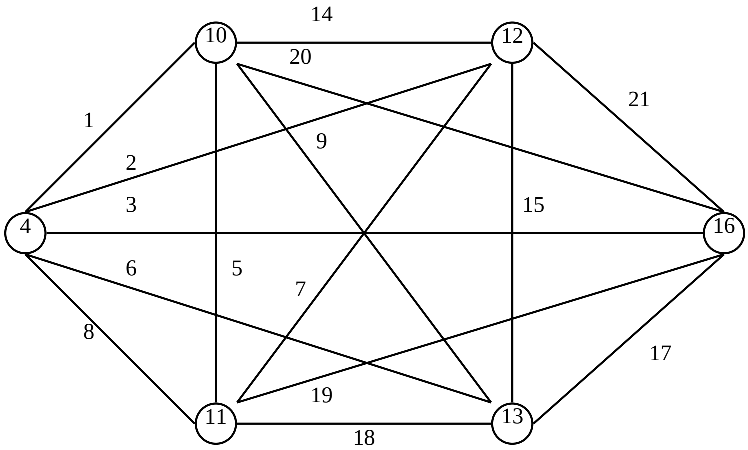


Figure 6.9: SVM - labeling of K_6, the complete graph of order 6.

6.9 Regular graphs of order 7

There are 5 regular graphs of order 7 that do not have any isolated and pendant vertex. They are $C_7, C_3 \cup C_4$, which are 2 - regulars, two non-isomorphic 4 - regular graphs and K_7, the complete graph which is 6 - regular. We have already proved that C_7 and $C_3 \cup C_4$ are SVM graphs. Let us investigate the SVM behaviour of the rest of graphs of order 7.

6.9.1 4 - regular graphs of order 7

We start with 4 - regular graphs of order 7. As in previous cases we can use the following hint that;

$$\sum_{v \in V(G)} f^v(v) = \frac{2 \times \sum_{e \in E(G)} f(e)}{4}$$

and, since $p + q = 7 + 14 = 21$, we have

$$\frac{(p+q)(p+q+1)}{2} = 231$$

$$\frac{3}{2} \times \sum_{e \in E(G)} f(e) = 231$$

$$\sum_{v \in V(G)} f^v(v) + \sum_{e \in E(G)} f(e) = 231$$

$$\sum_{e \in E(G)} f(e) = \frac{2 \times 231}{3} = 154$$

$$\sum_{v \in V(G)} f^v(v) = \frac{231}{3} = 77$$

So we partition the numbers up to 21 into two possible sets, having q and p elements respectively;

$$f(E) = \{1, 2, 4, 5, 6, 7, 8, 9, 13, 15, 16, 17, 18, 20, 21\}$$

$$f^v(V) = \{3, 8, 10, 11, 12, 12, 19\}$$

These two partitions give rise to two different labelings for the two non-isomorphic 4-regular graphs of order 7 as shown in Figure 6.10.

6.9.2 The complete graph K_7

Now we proceed to prove that K_7 is a SVM graph. K_7 being a 6 -regular graph of order 7 and each vertx is connected to every other vertex by a unique edge, we have to partition the numbers up to $\frac{n \times (n+1)}{2}$, since for a complete graph, K_n, $p + q = \frac{n \times (n+1)}{2}$.
i.e.,

$$\frac{7 \times 8}{2} = 28$$

The hint that we could use as in previous cases is that

$$\sum_{v \in V(G)} f^v(v) = \frac{2 \times \sum_{e \in E(G)} f(e)}{6}$$

and, since $p + q = 28$, we have

$$\frac{(p+q)(p+q+1)}{2} = 406$$

$$\frac{4}{3} \times \sum_{e \in E(G)} f(e) = 406$$

$$\sum_{v \in V(G)} f^v(v) + \sum_{e \in E(G)} f(e) = 406$$

$$\sum_{e \in E(G)} f(e) = \frac{406 \times 3}{4} = 101.5$$

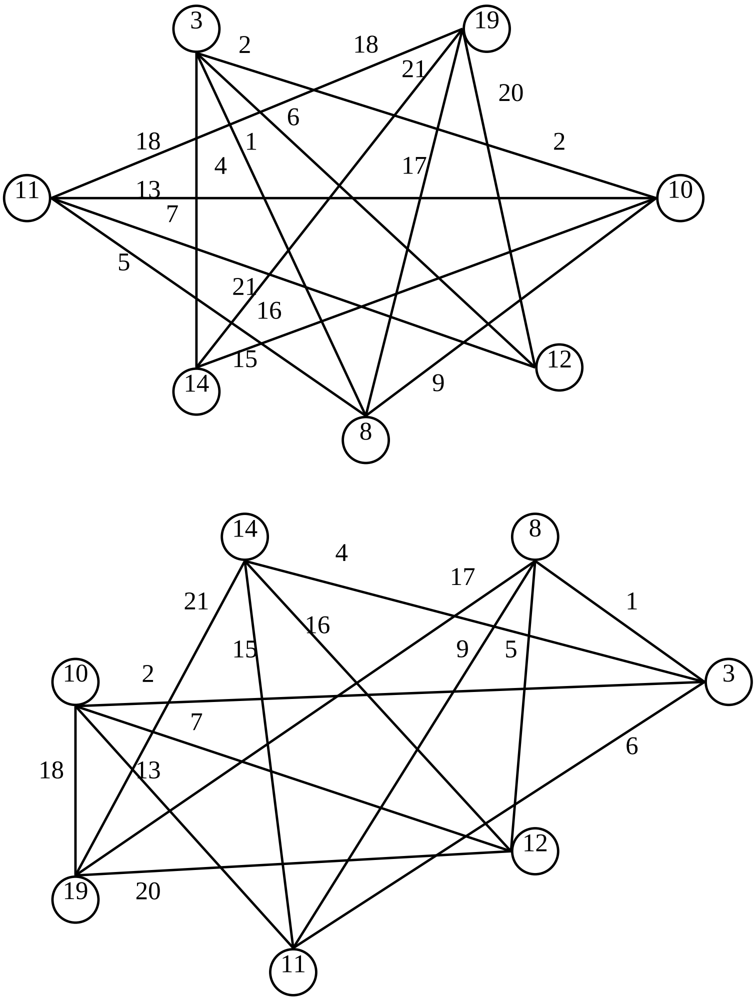


Figure 6.10: Two non-isomorphic 4 -regular graphs of order 7 are SVM labeled

$$\sum_{v \in V(G)} f^v(v) = \frac{406}{4} = 304.5$$

For our convenience, we take this as

$$\sum_{e \in E(G)} f(e) = 102$$

$$\sum_{v \in V(G)} f^v(v) = 304$$

Based on this, we obtain the following partitions of the numbers upto 28

$$f(E) = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 14, 17, 18, 20, 21, 22, 24, 25, 26, 27, 28\}$$

$$f^v(V) = \{4, 12, 13, 15, 16, 19, 23\}$$

having q and p elements respectively.

Careful distribution of these numbers as various edge as well as vertex labels, keeping the facts mentioned in earlier cases, we obtain the SVM labeling of K_7 as shown in Figure 6.11..

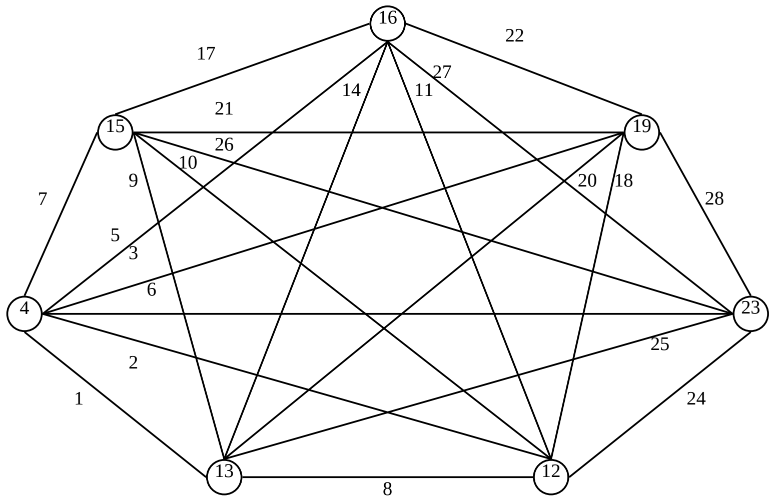


Figure 6.11: SVM - labeling of K_7

6.10 Super Vertex Mean Graphs of Order ≤ 5

Theorem 6.10.1. *All the graphs of order ≤ 5 having no isolated or pendant vertex are Super Vertex Mean graphs, C_4 being the only exception.*

We have so far proved that all the complete and regular graphs, (except C_4) of order up to 7, are SVM graphs and graphs containing any isolated or pendant vertex are not SVM graphs. In this section we examine all other graphs of order ≤ 5 and do not fall into the above category of graphs.

6.10.1 Of the order 4

There are 3 graphs with $d(v) \geq 2$ of order 4, out of which a graph with 5 edges fulfill our requirement and so we examine its SVM - behaviour and find that it is a SVM - graph. Its labeling is given in Figure 6.12.

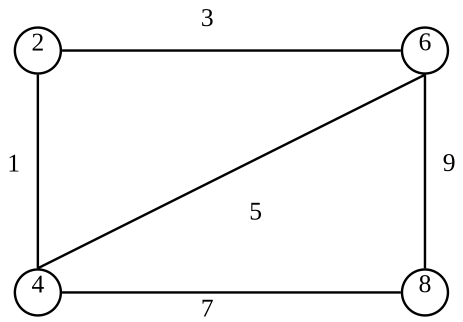


Figure 6.12: SVM labeling of a graph with 5 edges and of order 4

6.10.2 Of the order 5

Of the order 5, there are altogether 10 non-isomorphic graphs with $d(v) \geq 2$. Among those, the SVM - nature of 8 more graphs needs to be investigated for our present study. We have found that they are all SVM - graphs as shown in Figure 6.13..

6.11 Conclusion

We conclude by stating that all the r - regular graphs of order ≤ 7 and all graphs having no isolated or pendant vertex and order ≤ 5, excluding C_4 are SVM - graphs.

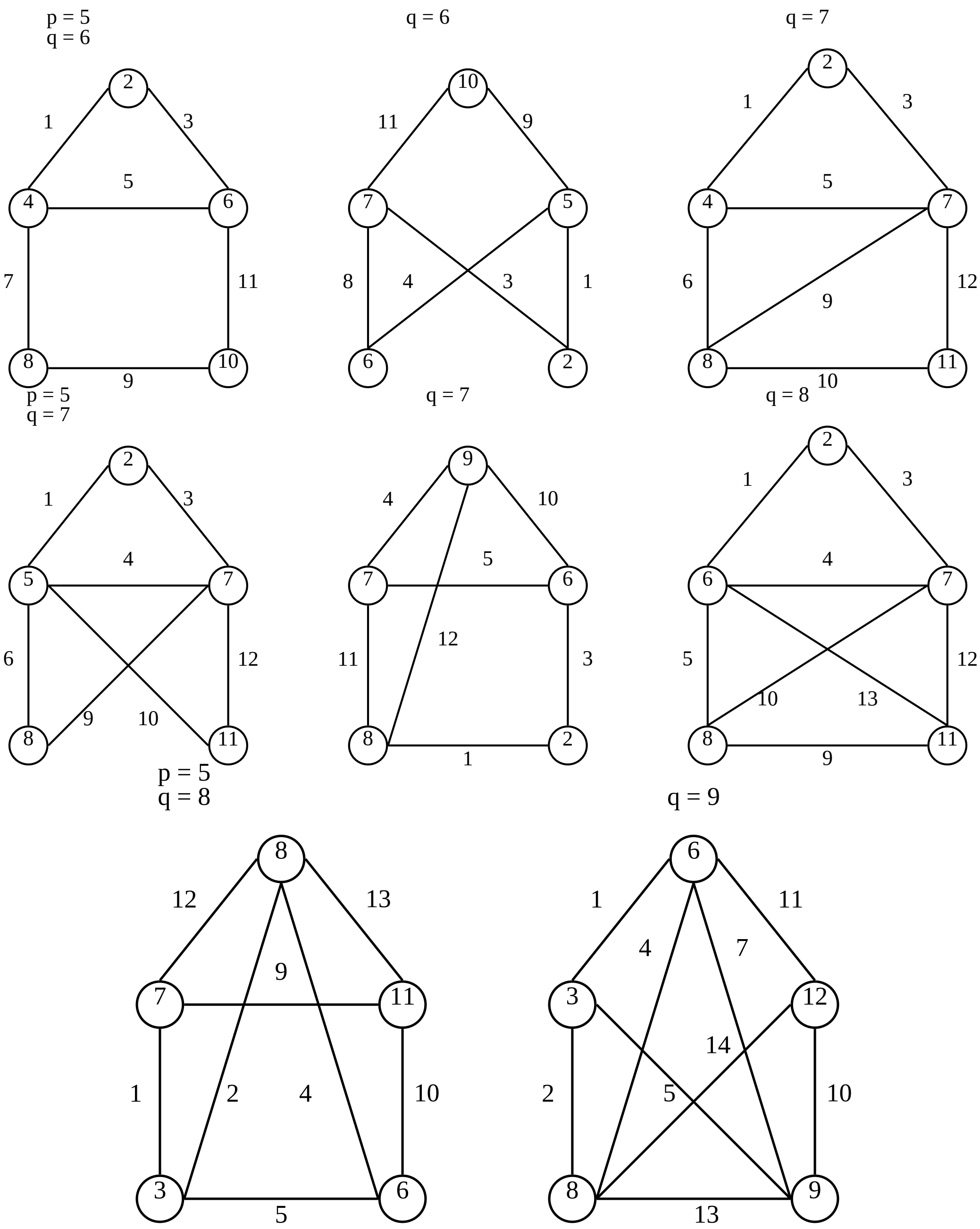


Figure 6.13: 8 non-isomorphic Graphs of order 5 are SVM.

Chapter 7

On Construction of SVM - Graphs

In this chapter we will show that disjoint union of all SVM - graphs, especially cycles, including mC_4, and those containing C_4 in it, is SVM - graph, even though C_4 is not an SVM - graph. We also construct some new SVM - graphs by joining two vertices of a cycle C_n by a chord. We also discuss the SVM - behaviour of P_n^2 graphs, though P_n is not an SVM - graph.

7.1 A Few Known Results

1. **If G is an SVM - graph, so is mG and if G_1 and G_2 are SVM - graphs, so is $G_1 \cup G_2$.**

2. **Although C_4 is not a SVM - graph, $2C_4$ and $C_3 \cup C_4$ are SVM - graphs.**

3. $C_3 \cup C_m$ **is SVM for all $m \geq 3$ including $m = 4$.**

4. mC_4 **is a SVM - graph for all even $m \geq 2$.**

5. **Disjoint union of any number of cycles of any length, except C_4 is a SVM - graph.**

6. **When the disjoint union of any number of cycles of any length contains C_4, it is a SVM - graph when,**

    - there are even number of C_4 in the union, or

- there exists at least one C_3 in the union.

7.2 Disjoint Union of Cycles as SVM - Graph

we proceed to prove that mC_4 for all $m \geq 2$ is a Super Vertex Mean Graph.

7.2.1 mC_4 for all $m \geq 2$ as SVM - graph

Theorem 7.2.1. *mC_4 for all $m \geq 2$ is a Super Vertex Mean Graph*

Proof. We know that mC_4 is a SVM - graph for all even m [Result 4]. So it is enough to prove that mC_4 is a SVM - graph for all odd m. For this we prove that $3C_4$ is a SVM - graph. Let C_4, C'_4 and C''_4 be 3 cycles of length 4.
Let $E(C_4) = \{e_1, e_2, e_3, e_4\}, E(C'_4) = \{e'_1, e'_2, e'_3, e'_4\}$ and $E(C''_4) = \{e''_1, e''_2, e''_3, e''_4\}$
Define $f : E(C_4 \cup C'_4 \cup C''_4) \to \{1, 2, 3, ..., 23, 24\}$ as follows:

$$f(e_1) = 1, f(e_2) = 3, f(e_3) = 5, f(e_4) = 11,$$

$$f(e'_1) = 7, f(e'_2) = 10, f(e'_3) = 16, f(e'_4) = 21,$$

$$f(e''_1) = 12, f(e''_2) = 17, f(e''_3) = 22, f(e''_4) = 24.$$

And the induced vertex label set is $\{2, 4, 6, 8, 9, 13, 14, 15, 18, 19, 20, 23\}$.
It can be easily verified that f is a super vertex mean labeling.
Since disjoint union of any SVM - graphs is an SVM - graph, we have the result. □

7.2.2 $C_m \cup C_n$ for all $m \geq 3$ and $n \geq 3$ as SVM - graph

Theorem 7.2.2. *The graph $C_m \cup C_n$ is a SVM - Graph for all $m \geq 3$ and $n \geq 3$.*

Proof. All cycles, except C_4 and their disjoint unions are SVM - graphs [Result 5]. So we know that the graph $C_m \cup C_n$ is a SVM - Graph excepting the case where the union contains C_m or C_n where either m or n is equal to 4.

Let $m = 4$ and $E(C_m) = \{e_1, e_2, e_3, e_4\}$ and $V(C_m) = \{v_1, v_2, v_3, v_4\}$ where $v_i = e_i e_{i+1}, 1 \leq i \leq 3$ and $v_4 = e_4 e_1$

Let $E(C_n) = \{e'_1, e'_2, \cdots, e'_n\}, V(C'_n) = \{v'_1, v'_2, v'_3, \cdots, v'_n\}$ such that $v'_i = e'_i e'_{i+1}, 1 \leq i \leq n-1$ and $v_n = e'_n e'_1$

Here $p = n + 4$ & $q = n + 4$ and so $p + q = 2n + 8$.

We consider the following two cases;

Case 1. When n is even.

If $n = 4$, then we know that $2C_4$ is a SVM - graph. Therefore let $n \geq 6$.

Define $f : E(C_n \cup C_4) \to \{1, 2, 3, \cdots, 2n + 8\}$ by

$$f(e'_i) = \begin{cases} 2i - 1, & \text{if } 1 \leq i \leq \frac{n}{2} + 2, \\ 2i + 8, & \text{if } \frac{n}{2} + 3 \leq i \leq n. \end{cases}$$

$$f(e_1) = n + 4, f(e_2) = n + 7, f(e_3) = n + 13, f(e_4) = n + 11.$$

Clearly f is an injective function and

$$f(E) \cup f(V) = \{1, 2, 3, \cdots, 2n + 8\}.$$

Therefore f is a SVM - labeling of $C_4 \cup C_n$, where n is even.

Case 2. When n is odd.

We know that when $n = 3$, $C_4 \cup C_3$ is a SVM - graph. So let $n \geq 5$.

Define $f : E(C_4 \cup C_n) \to \{1, 2, 3, \cdots, 2n + 8\}$ by

$$f(e_1) = 1, f(e_2) = 3, f(e_3) = 5, f(e_4) = 11.$$

$$f(e'_i) = \begin{cases} 7, & \text{if } i = 1, \\ 10, & \text{if } i = 2, \\ 8 + 2i, & \text{if } 3 \leq i \leq 5, \\ 4i - 2, & \text{if } 6 \leq i \leq \frac{n+5}{2}, \\ 4n + 19 - 4i, & \text{if } \frac{n+7}{2} \leq i \leq n. \end{cases}$$

Clearly f is an injective function and

$$f(E) \cup f(V) = \{1, 2, 3, \cdots, 2n + 8\}.$$

Therefore f is a SVM - labeling of $C_4 \cup C_n$, where n is odd.
Hence $C_m \cup C_n$ is a SVM - Graph for all $m \geq 3$ and $n \geq 3$. □

Corollary 7.2.3. *Disjoint union of any number of cycles of any length is a SVM - graph, except the fact that C_4 is not a SVM - graph.*

Proof. The result is obtained from the above two theorems. □

7.2.3 SVM - labeling method for union of SVM - graphs

Let $G_1, G_2, \cdots, G_m$ be m SVM graphs with SVM - labelings $f_1, f_2, \cdots, f_m$ respectively. By the above theorem we know that $G_1 \cup G_2 \cup \cdots \cup G_n$ is a SVM - graph. We discuss the method of labeling this new graph.

Let $G_1 = (p_1, q_1), G_2 = (p_2, q_2), \cdots G_m = (p_m, q_m)$ be the m SVM graphs. Then $G_1 \cup G_2 \cup \cdots \cup G_n$ has $p_1 + p_2 + \cdots + p_m$ vertices and $q_1 + q_2 + \cdots + q_m$ edges.

Let $e_{1i}, 1 \leq i \leq q_1, e_{2i}, 1 \leq i \leq q_2, \cdots, e_{mi}, 1 \leq i \leq q_m$ and $v_{1i}, 1 \leq i \leq p_1, v_{2i}, 1 \leq i \leq p_2, \cdots, v_{mi}, 1 \leq i \leq p_m$ be the edges and vertices of the graph $G_1, G_2, \cdots, G_m$ respectively.

Define $g : E(G_1 \cup G_2 \cup \ldots, G_n) \rightarrow \{1, 2, \cdots, p_1 + p_2 + \cdots + q_1 + q_2 + \cdots + q_m\}$ as follows:

$$g(e_{1i}) = f_1(e_{1i})$$

$$g(e_{2i}) = p_1 + q_1 + f_2(e_{2i})$$

$$g(e_{3i}) = p_1 + q_1 + p_2 + q_2 + f_3(e_{3i})$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$g(e_{mi}) = p_1 + q_1 + p_2 + q_2 + p_3 + q_3 + \cdots + p_m + q_m + f_m(e_{mi})$$

Then the induced vertex labels will be as follows:

$$g^v(v_{1i}) = f_1^v(v_{1i})$$

$$g^v(v_{2i}) = p_1 + q_1 + f_2^v(v_{2i})$$

$$g^v(v_{3i}) = p_1 + q_1 + p_2 + q_2 + f_3^v(v_{3i})$$

$$\ldots\ldots\ldots$$

$$\ldots\ldots\ldots$$

$$g^v(v_{mi}) = p_1 + q_1 + p_2 + q_2 + p_3 + q_3 + \cdots + p_m + q_m + f_m^v(v_{mi})$$

7.3 $P_n^2, n \geq 3$ as SVM - graph

Theorem 7.3.1. *The graph $P_n^2, n \geq 3$ is a SVM - graph.*

Proof. Let P_n be a path $u_1u_2\cdots u_n, n \geq 3$. Then P_n^2 is a graph with the edge set $E = \{u_1u_2, u_2u_3, \cdots, u_{n-1}u_n, u_1u_3, u_2u_4, \cdots, u_{n-2}u_n\}$. Clearly P_n^2 has n vertices and $2n-3$ edges.

Obviously P_3^2 is C_3 whose labeling we have discussed already. So let $n \geq 4$. Define $f : E \to \{1, 2, 3, \cdots, 3n-3\}$ as follows:

$$f(u_iu_{i+1}) = \begin{cases} 1, & \text{if } i = 1, \\ 5, & \text{if } i = 2, \\ 3i-1, & \text{if } 3 \leq i \leq n-2 \text{ and } i \text{ is odd}, \\ 3i-2, & \text{if } 4 \leq i \leq n-2 \text{ and } i \text{ is even}, \\ 3n-3, & \text{if } i = n-1. \end{cases}$$

$$f(u_iu_{i+2}) = \begin{cases} 3, & \text{if } i = 1, \\ 3i, & \text{if } 2 \leq i \leq n-3 \text{ and } i \text{ is even}, \\ 3i+2 & \text{if } 3 \leq i \leq n-3 \text{ and } i \text{ is odd}, \\ 3n-5, & \text{if } i = n-2. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2, & \text{if } i = 1, \\ 4, & \text{if } i = 2, \\ 3i - 2, & \text{if } 3 \leq i \leq n - 2 \text{ and } i \text{ is odd}, \\ 3i, & \text{if } 4 \leq i \leq n - 2 \text{ and } i \text{ is even}, \\ 3n - 6, & \text{if } i = n - 1, \\ 3n - 4, & \text{if } i = n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 3n - 3\}$. □

7.4 Cycle related Graphs

In a graph G, the distance between two vertices u and v denoted by $d_G(u, v)$ is the length of the shortest path joining u and v. Let H be a subgraph of G. Then $d_H(u, v)$ denotes the distance between u and v in H. In this section we find the SVM - labeling of C_n together with a chord uv such that $d_{C_n}(u, v) = 2, 3, 4, 5, 6$ or 7

7.4.1 Cycle with a Chord connecting arc of distance 2

Theorem 7.4.1. *Let C_n be a cycle of length $n \geq 4$ and let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$ such that $d_{C_n}(u, v) = 2$, where $u, v \in V(C_n)$. Then G is a SVM - graph.*

Proof. Let $C_n, n \geq 4$ be a cycle $u_1 u_2 \cdots u_n u_1$ and let $u = u_2$ and $v = u_n$. Then $d_{C_n}(u, v) = 2$.

Let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$. Then G has $n + 1$ edges and n vertices, making a total of $2n + 1$ elements.

Let the edges of G be such that

$$e_i = u_i u_{i+1}, 1 \leq i \leq n-1, e_n = u_n u_1 \ \& \ e_{n+1} = uv = u_2 u_n$$

The SVM labelings of graphs thus obtained from C_4, C_5, C_6, C_7 and C_{10} are shown in the following Figure 7.1.

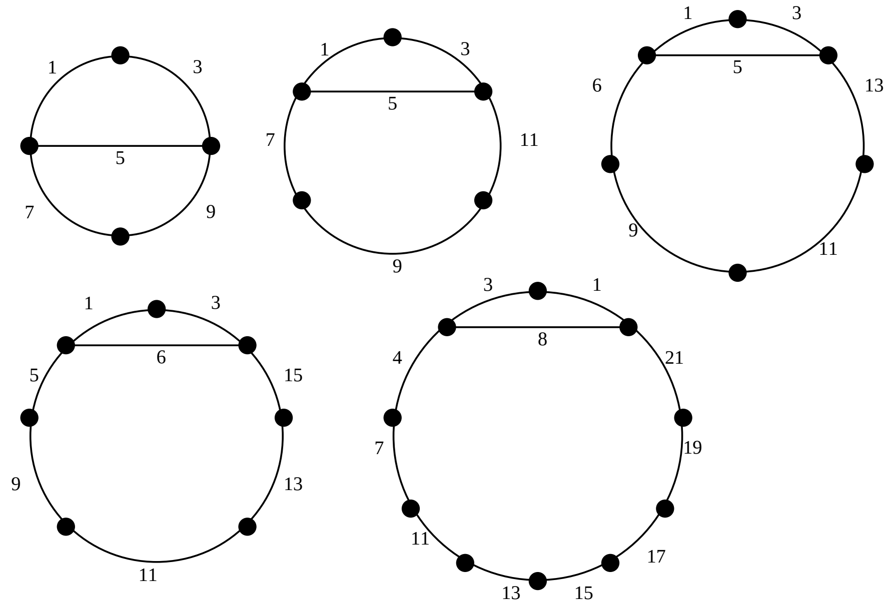


Figure 7.1: SVM - Labeling of $C_n \cup \{uv\}, n = 4, 5, 6, 7, 10$ respectively.

Now we continue to discuss Super Vertex Meanness of rest of the graphs obtained in the following two cases.

Case 1: When $n \equiv 0$ or $2(mod\ 3), n \geq 8$

Define $f : E(G) \to \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 1, & \text{if } i = 1, \\ 5, & \text{if } i = 2, \\ 2i+2, & \text{if } 3 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i+3, & \text{if } \lceil \frac{n}{3} \rceil + 1 \leq i \leq n-1, \\ 3, & \text{if } i = n, \\ 6, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2, & \text{if } i = 1, \\ 4, & \text{if } i = 2, \\ 2i+1, & \text{if } 3 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i+2, & \text{if } \lceil \frac{n}{3} \rceil + 1 \leq i \leq n-1, \\ 2\lceil \frac{n}{3} \rceil + 3, & \text{if } i = n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n+1\}$.

Case 2: When $n \equiv 1(mod\, 3), n \geq 13$

Define $f : E(G) \to \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 1, & \text{if } i = 1, \\ 4, & \text{if } i = 2, \\ 7, & \text{if } i = 3, \\ 9, & \text{if } i = 4, \\ 2i+2, & \text{if } 5 \leq i \leq \lceil \frac{n}{3} \rceil, \end{cases}$$

$$f(e_i) = \begin{cases} 2i+3, & \text{if } \lceil \frac{n}{3} \rceil + 1 \leq i \leq n-1, \\ 3, & \text{if } i = n, \\ 10, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2, & \text{if } i = 1, \\ 5, & \text{if } i = 2, \\ 6, & \text{if } i = 3, \\ 8, & \text{if } i = 4, \\ 2i+1, & \text{if } 5 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i+2, & \text{if } \lceil \frac{n}{3} \rceil + 1 \leq i \leq n-1, \\ 2\lceil \frac{n}{3} \rceil + 3, & \text{if } i = n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n+1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph. □

7.4.2 Cycle with a Chord connecting arc of distance 3

Theorem 7.4.2. *Let C_n be a cycle of length $n \geq 6$ and let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$ such that $d_{C_n}(u, v) = 3$, where $u, v \in V(C_n)$. Then G is a SVM - graph.*

Proof. Let $C_n, n \geq 6$ be a cycle $u_1u_2\cdots u_nu_1$ and let $u = u_1$ and $v = u_{n-2}$. Then $d_{C_n}(u, v) = 3$.

Let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$. Then G has $n+1$ edges and n vertices, making a total of $2n+1$ elements. Let the edges of G be such that

$$e_i = u_iu_{i+1}, 1 \leq i \leq n-1, e_n = u_nu_1 \;\&\; e_{n+1} = uv = u_1u_{n-2}$$

We prove the theorem in the following three cases.

Case 1: When $n \equiv 0(mod\, 3), n \geq 6$

The SVM labeling of graph obtained from C_6 is shown in the following Figure 7.2.

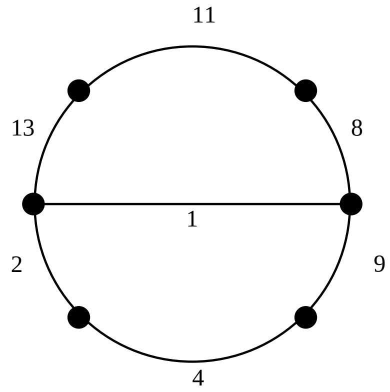


Figure 7.2: Super Vertex Mean Labeling of $C_6 \cup \{uv\}$.

When $n \geq 9$ define $f : E(G) \rightarrow \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 2i, & \text{if } 1 \leq i \leq \frac{n}{3}, \\ 2i+1, & \text{if } \frac{n}{3}+1 \leq i \leq \frac{2}{3}n-2, \\ 2i+2, & \text{if } \frac{2}{3}n-1 \leq i \leq n-3, \\ 2i+1, & \text{if } n-2 \leq i \leq n, \\ 1, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} \frac{2}{3}n+1, & \text{if } i = 1, \\ 2i-1, & \text{if } 2 \leq i \leq \frac{n}{3}, \\ 2i, & \text{if } \frac{n}{3}+1 \leq i \leq \frac{2}{3}n-2, \\ 2i+1, & \text{if } \frac{2}{3}n-1 \leq i \leq n-3, \\ 4\frac{n}{3}-2, & \text{if } i = n-2, \\ 2i, & \text{if } n-1 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n+1\}$.

Case 2: When $n \equiv 1(mod\, 3), n \geq 7$

The SVM labeling of graph obtained from C_7 is shown in the following Figure 7.3.

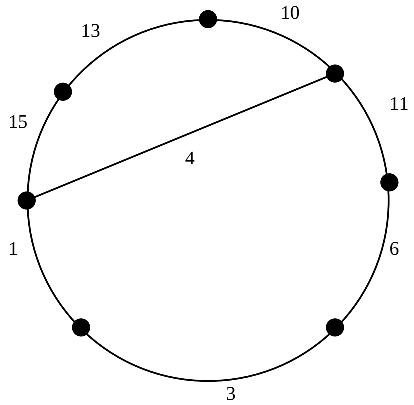


Figure 7.3: Super Vertex Mean Labeling of $C_7 \cup \{uv\}$.

For $n \geq 10$ define $f : E(G) \rightarrow \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 2i-1, & \text{if } 1 \leq i \leq 2, \\ 2i, & \text{if } 3 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i+1, & \text{if } \lceil \frac{n}{3} \rceil + 1 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 3, \\ 2i+2, & \text{if } 2\lceil \frac{n}{3} \rceil - 2 \ \leq i \leq n-3, \\ 2i+1, & \text{if } n-2 \leq i \leq n, \\ 4, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 1, & \text{if } i = 1, \\ 2, & \text{if } i = 2, \\ 2i - 1, & \text{if } 3 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i, & \text{if } \lceil \frac{n}{3} \rceil + 1 \leq i \leq 2\lceil \frac{n}{3} \rceil - 3, \\ 2i + 1, & \text{if } 2\lceil \frac{n}{3} \rceil - 2 \leq i \leq n - 3, \\ 4\lceil \frac{n}{3} \rceil - 4, & \text{if } i = n - 2, \\ 2i, & \text{if } n - 1 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n + 1\}$. Since f is a Super Vertex Mean labeling, G is a SVM graph.

Case 3: When $n \equiv 2(mod\, 3), n \geq 8$

Define $f : E(G) \to \{1, 2, 3, \cdots, 2n + 1\}$ as follows,

$$f(e_i) = \begin{cases} 2i - 1, & \text{if } 1 \leq i \leq 2, \\ 2i, & \text{if } 3 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i + 1, & \text{if } \lceil \frac{n}{3} \rceil + 1 \leq i \leq 2\lceil \frac{n}{3} \rceil - 2, \\ 2i + 2, & \text{if } 2\lceil \frac{n}{3} \rceil - 1 \leq i \leq n - 3, \\ 2i + 1, & \text{if } n - 2 \leq i \leq n, \\ 4, & \text{if } i = n + 1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 1, & \text{if } i = 1, \\ 2, & \text{if } i = 2, \\ 2i - 1, & \text{if } 3 \le i \le \lceil \frac{n}{3} \rceil, \\ 2i, & \text{if } \lceil \frac{n}{3} \rceil + 1 \le i \le 2\lceil \frac{n}{3} \rceil - 2, \\ 2i + 1, & \text{if } 2\lceil \frac{n}{3} \rceil - 1 \le i \le n - 3, \\ 4\lceil \frac{n}{3} \rceil - 2, & \text{if } i = n - 2, \\ 2i, & \text{if } n - 1 \le i \le n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n + 1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph. □

7.4.3 Cycle with a Chord connecting arc of distance 4

Theorem 7.4.3. *Let C_n be a cycle of length $n \ge 8$ and let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$ such that $d_{C_n}(u, v) = 4$, where $u, v \in V(C_n)$. Then G is a SVM - graph.*

Proof. Let $C_n, n \ge 8$ be a cycle $u_1u_2 \cdots u_nu_1$ and let $u = u_1$ and $v = u_{n-3}$. Then $d_{C_n}(u, v) = 4$.

Let the edges of G be such that

$$e_i = u_iu_{i+1}, 1 \le i \le n - 1, e_n = u_nu_1 \ \& \ e_{n+1} = uv = u_1u_{n-3}$$

Let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$. Then G has $n + 1$ edges and n vertices, making a total of $2n + 1$ elements.

We prove the theorem in the following three cases.

Case 1: When $n \equiv 0 (mod\ 3), n \ge 9$

The SVM labeling of graph obtained from C_9 is shown in the following Figure 7.4.

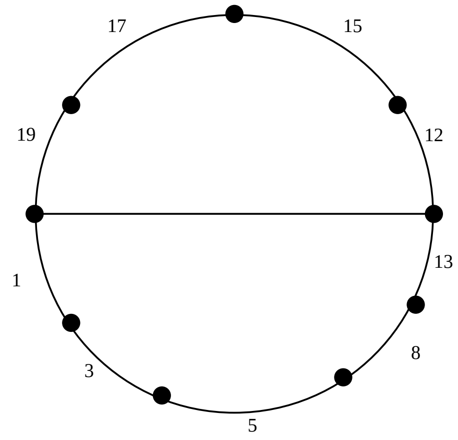


Figure 7.4: Super Vertex Mean Labeling of $C_9 \cup \{uv\}$.

When $n \geq 12$ define $f : E(G) \rightarrow \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 2i-1, & \text{if } 1 \leq i \leq 3, \\ 2i, & \text{if } 4 \leq i \leq \frac{n}{3}+1, \\ 2i+1, & \text{if } \frac{n}{3}+2 \leq i \leq \frac{2}{3}n-2, \\ 2i+2, & \text{if } \frac{2}{3}n-1 \leq i \leq n-4, \\ 2i+1, & \text{if } n-3 \leq i \leq n, \\ 6, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\frac{n}{3}+3, & \text{if } i = 1, \\ 2i, & \text{if } 2 \leq i \leq 3, \\ 2i-1, & \text{if } 4 \leq i \leq \frac{n}{3}+1, \\ 2i, & \text{if } \frac{n}{3}+2 \leq i \leq \frac{2}{3}n-2, \\ 2i+1, & \text{if } 2\frac{n}{3}-1 \leq i \leq n-4, \\ 4\frac{n}{3}-2, & \text{if } i = n-3, \\ 2i, & \text{if } n-2 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n+1\}$. Since f is a Super Vertex Mean labeling, G is a SVM graph.

Case 2: When $n \equiv 1(mod\,3), n \geq 10$

Define $f : E(G) \to \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 2i-1, & \text{if } 1 \leq i \leq 3, \\ 2i, & \text{if } 4 \leq i \leq \lfloor \frac{n}{3} \rfloor + 1, \\ 2i+1, & \text{if } \lfloor \frac{n}{3} \rfloor + 2 \ \leq i \leq 2\lfloor \frac{n}{3} \rfloor - 1, \\ 2i+2, & \text{if } 2\lfloor \frac{n}{3} \rfloor \ \leq i \leq n-4, \\ 2i+1, & \text{if } n-3 \leq i \leq n, \\ 6, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\lfloor \frac{n}{3} \rfloor + 3, & \text{if } i = 1, \\ 2i, & \text{if } 2 \leq i \leq 3, \\ 2i-1, & \text{if } 4 \leq i \leq \lfloor \frac{n}{3} \rfloor + 1, \\ 2i, & \text{if } \lfloor \frac{n}{3} \rfloor + 2 \ \leq i \leq 2\lfloor \frac{n}{3} \rfloor - 1, \\ 2i+1, & \text{if } 2\lfloor \frac{n}{3} \rfloor \ \leq i \leq n-4, \\ 4\lfloor \frac{n}{3} \rfloor, & \text{if } i = n-3, \\ 2i, & \text{if } n-2 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n+1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph.

Case 3: When $n \equiv 2(mod\,3), n \geq 8$

The SVM - labeling of graph obtained from C_8 is shown in the following Figure 7.5.

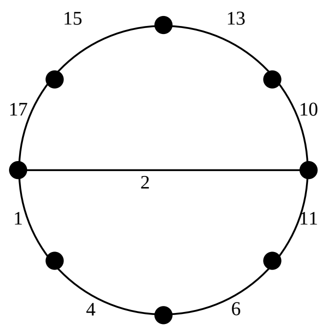


Figure 7.5: Super Vertex Mean Labeling of $C_8 \cup \{uv\}$.

For $n \geq 11$ define $f : E(G) \to \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 1, & \text{if } i = 1, \\ 2i, & \text{if } 2 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i+1, & \text{if } \lceil \frac{n}{3} \rceil + 1 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 3, \\ 2i+2, & \text{if } 2\lceil \frac{n}{3} \rceil - 2 \ \leq i \leq n-4, \\ 2i+1, & \text{if } n-3 \leq i \leq n, \\ 2, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 1, & \text{if } i = 1, \\ 2i-1, & \text{if } 2 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i, & \text{if } \lceil \frac{n}{3} \rceil + 1 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 3, \\ 2i+1, & \text{if } 2\lceil \frac{n}{3} \rceil - 2 \ \leq i \leq n-4, \\ 4\lceil \frac{n}{3} \rceil - 4, & \text{if } i = n-3, \\ 2i, & \text{if } n-2 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n+1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph. □

7.4.4 Cycle with a Chord connecting arc of distance 5

Theorem 7.4.4. *Let C_n be a cycle of length $n \geq 10$ and let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$ such that $d_{C_n}(u, v) = 5$, where $u, v \in V(C_n)$. Then G is a SVM - graph.*

Proof. Let $C_n, n \geq 10$ be a cycle $u_1u_2 \cdots u_nu_1$ and let $u = u_1$ and $v = u_{n-4}$. Then $d_{C_n}(u, v) = 5$.
Let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$.
Then G has $n+1$ edges and n vertices, making a total of $2n+1$ elements.
Let the edges of G be such that

$$e_i = u_iu_{i+1}, 1 \leq i \leq n-1, e_n = u_nu_1 \ \& \ e_{n+1} = uv = u_1u_{n-4}$$

We prove the theorem in the following three cases.
Case 1: When $n \equiv 0(mod\, 3), n \geq 12$
Define $f : E(G) \to \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 1, & \text{if } i = 1, \\ 2i, & \text{if } 2 \leq i \leq \frac{n}{3}, \\ 2i+1, & \text{if } \frac{n}{3}+1 \ \leq i \leq \frac{2}{3}n-3, \\ 2i+2, & \text{if } \frac{2}{3}n-2 \ \leq i \leq n-5, \\ 2i+1, & \text{if } n-4 \leq i \leq n, \\ 2, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\frac{n}{3}+1, & \text{if } i=1, \\ 2i-1, & \text{if } 2 \le i \le \frac{n}{3}, \\ 2i, & \text{if } \frac{n}{3}+1 \ \le i \le \frac{2}{3}n-3, \\ 2i+1, & \text{if } \frac{2}{3}n-2 \ \le i \le n-5, \\ 4\frac{n}{3}-4, & \text{if } i=n-4, \\ 2i, & \text{if } n-3 \le i \le n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1,2,3,\cdots,2n+1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph.

Case 2: When $n \equiv 1(mod\, 3), n \ge 10$

The SVM labeling of graph obtained from C_{10} is shown in the following Figure 7.6.

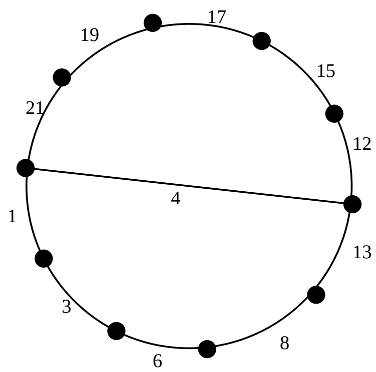


Figure 7.6: Super Vertex Mean Labeling of $C_{10} \cup \{uv\}$.

For $n \ge 13$ define $f : E(G) \to \{1,2,3,\cdots,2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 2i-1, & \text{if } 1 \le i \le 2, \\ 2i, & \text{if } 3 \le i \le \lceil \frac{n}{3} \rceil, \\ 2i+1, & \text{if } \lceil \frac{n}{3} \rceil +1 \ \le i \le 2\lceil \frac{n}{3} \rceil - 4, \end{cases}$$

$$f(e_i) = \begin{cases} 2i+2, & \text{if } 2\lceil \frac{n}{3} \rceil - 3 \ \leq i \leq n-5, \\ 2i+1, & \text{if } n-4 \leq i \leq n, \\ 4, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 1, & \text{if } i = 1, \\ 2, & \text{if } i = 2, \\ 2i-1, & \text{if } 3 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i, & \text{if } \lceil \frac{n}{3} \rceil + 1 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 4, \\ 2i+1, & \text{if } 2\lceil \frac{n}{3} \rceil - 3 \ \leq i \leq n-5, \\ 4\lceil \frac{n}{3} \rceil - 6, & \text{if } i = n-4, \\ 2i, & \text{if } n-3 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n+1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph.

Case 3: When $n \equiv 2(mod\ 3), n \geq 11$

The SVM - labeling of graph obtained from C_{11} is shown in the following Figure 7.7.

For $n \geq 14$ define $f : E(G) \rightarrow \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 2i-1, & \text{if } 1 \leq i \leq 4, \\ 2i, & \text{if } 5 \leq i \leq \lceil \frac{n}{3} \rceil + 1, \\ 2i+1, & \text{if } \lceil \frac{n}{3} \rceil + 2 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 3, \\ 2i+2, & \text{if } 2\lceil \frac{n}{3} \rceil - 2 \ \leq i \leq n-5, \\ 2i+1, & \text{if } n-4 \leq i \leq n, \\ 8, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

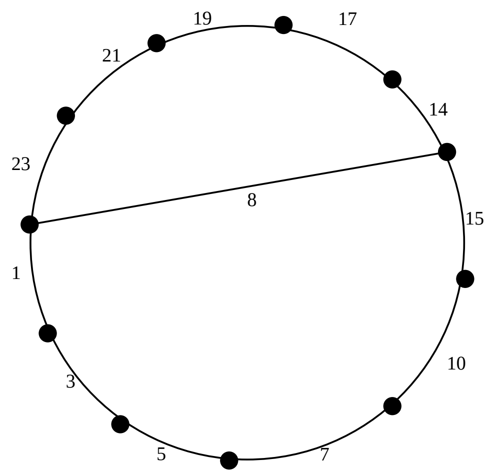


Figure 7.7: Super Vertex Mean Labeling of $C_{11} \cup \{uv\}$.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 3, & \text{if } i = 1, \\ 2i-2, & \text{if } 2 \leq i \leq 4, \\ 2i-1, & \text{if } 5 \leq i \leq \lceil \frac{n}{3} \rceil + 1, \\ 2i, & \text{if } \lceil \frac{n}{3} \rceil + 2 \leq i \leq 2\lceil \frac{n}{3} \rceil - 3, \\ 2i+1, & \text{if } 2\lceil \frac{n}{3} \rceil - 2 \leq i \leq n-5, \\ 4\lceil \frac{n}{3} \rceil - 4, & \text{if } i = n-4, \\ 2i, & \text{if } n-3 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n+1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph. □

7.4.5 Cycle with a Chord connecting arc of distance 6

Theorem 7.4.5. *Let C_n be a cycle of length $n \geq 12$ and let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$ such that $d_{C_n}(u, v) = 6$, where $u, v \in V(C_n)$. Then G is a SVM - graph.*

Proof. Let $C_n, n \geq 12$ be a cycle $u_1u_2\cdots u_nu_1$ and let $u = u_1$ and $v = u_{n-5}$. Then $d_{C_n}(u,v) = 6$.

Let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$. Then G has $n+1$ edges and n vertices, making a total of $2n+1$ elements.
Let the edges of G be such that

$$e_i = u_iu_{i+1}, 1 \leq i \leq n-1, e_n = u_nu_1 \;\&\; e_{n+1} = uv = u_1u_{n-5}$$

We prove the theorem in the following three cases.

Case 1: When $n \equiv 0(mod\, 3), n \geq 12$

The SVM labeling of graph obtained from C_{12} is shown in the following Figure 7.8.

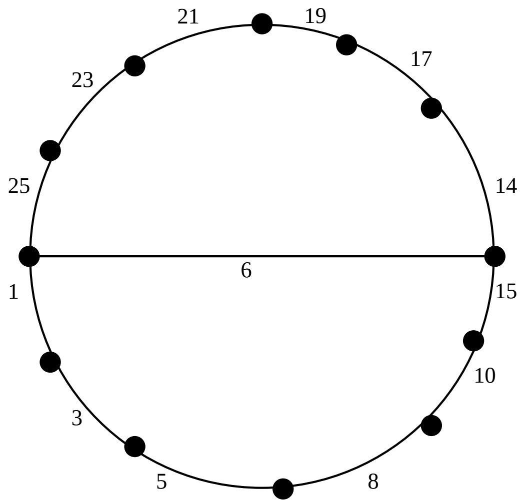


Figure 7.8: Super Vertex Mean Labeling of $C_{12} \cup \{uv\}$.

For $n \geq 15$ define $f : E(G) \to \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 2i-1, & \text{if } 1 \leq i \leq 3, \\ 2i, & \text{if } 4 \leq i \leq \frac{n}{3}+1, \\ 2i+1, & \text{if } \frac{n}{3}+2 \leq i \leq \frac{2}{3}n-3, \\ 2i+2, & \text{if } \frac{2}{3}n-2 \leq i \leq n-6, \\ 2i+1, & \text{if } n-5 \leq i \leq n, \\ 6, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\frac{n}{3}+3, & \text{if } i=1, \\ 2i-2, & \text{if } 2 \le i \le 3, \\ 2i-1, & \text{if } 4 \le i \le \frac{n}{3}+1, \\ 2i, & \text{if } \frac{n}{3}+2 \le i \le \frac{2}{3}n-3, \\ 2i+1, & \text{if } \frac{2}{3}n-2 \ \le i \le n-6, \\ 4\frac{n}{3}-4, & \text{if } i=n-5, \\ 2i, & \text{if } n-4 \le i \le n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n+1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph.

Case 2: When $n \equiv 1(mod\, 3), n \ge 13$

The SVM - labeling of graph obtained from C_{13} is shown in the following Figure 7.9.

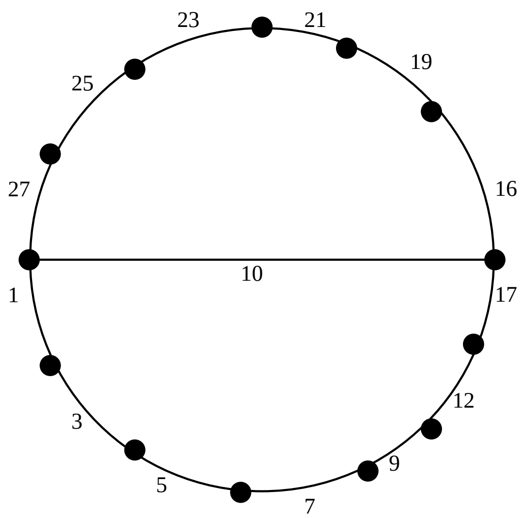


Figure 7.9: Super Vertex Mean Labeling of $C_{13} \cup \{uv\}$.

For $n \geq 16$ define $f : E(G) \rightarrow \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 2i-1, & \text{if } 1 \leq i \leq 5, \\ 2i, & \text{if } 6 \leq i \leq \lceil \frac{n}{3} \rceil + 1, \\ 2i+1, & \text{if } \lceil \frac{n}{3} \rceil + 2 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 4, \\ 2i+2, & \text{if } 2\lceil \frac{n}{3} \rceil - 3 \ \leq i \leq n-6, \\ 2i+1, & \text{if } n-5 \leq i \leq n, \\ 10, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 3, & \text{if } i = 1, \\ 2i-2, & \text{if } 2 \leq i \leq 5, \\ 2i-1, & \text{if } 6 \leq i \leq \lceil \frac{n}{3} \rceil + 1, \\ 2i, & \text{if } \lceil \frac{n}{3} \rceil + 2 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 4, \\ 2i+1, & \text{if } 2\lceil \frac{n}{3} \rceil - 3 \ \leq i \leq n-6, \\ 4\lceil \frac{n}{3} \rceil - 6, & \text{if } i = n-5, \\ 2i, & \text{if } n-4 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n+1\}$. Since f is a Super Vertex Mean labeling, G is a SVM graph.

Case 3: When $n \equiv 2(mod\ 3), n \geq 14$

Define $f : E(G) \to \{1, 2, 3, \cdots, 2n + 1\}$ as follows,

$$f(e_i) = \begin{cases} 2i - 1, & \text{if } 1 \leq i \leq 3, \\ 2i, & \text{if } 4 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i + 1, & \text{if } \lceil \frac{n}{3} \rceil + 1 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 4, \\ 2i + 2, & \text{if } 2\lceil \frac{n}{3} \rceil - 3 \ \leq i \leq n - 6, \\ 2i + 1, & \text{if } n - 5 \leq i \leq n, \\ 4, & \text{if } i = n + 1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 1, & \text{if } i = 1, \\ 2, & \text{if } i = 2, \\ 2i - 1, & \text{if } 3 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i - 1, & \text{if } \lceil \frac{n}{3} \rceil + 1 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 4, \\ 2i + 1, & \text{if } 2\lceil \frac{n}{3} \rceil - 3 \ \leq i \leq n - 6, \\ 4\lceil \frac{n}{3} \rceil - 6, & \text{if } i = n - 5, \\ 2i, & \text{if } n - 4 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n + 1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph. □

7.4.6 Cycle with a Chord connecting arc of distance 7

Theorem 7.4.6. *Let C_n be a cycle of length $n \geq 14$ and let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$ such that $d_{C_n}(u, v) = 7$, where $u, v \in V(C_n)$. Then G is a SVM - graph.*

Proof. Let $C_n, n \geq 14$ be a cycle $u_1u_2 \cdots u_nu_1$ and let $u = u_1$ and $v = u_{n-6}$. Then $d_{C_n}(u, v) = 7$.

Let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$. Then G has $n+1$ edges and n vertices, making a total of $2n+1$ elements.
Let the edges of G be such that

$$e_i = u_i u_{i+1}, 1 \leq i \leq n-1, e_n = u_n u_1 \;\&\; e_{n+1} = uv = u_1 u_{n-6}$$

We prove the theorem in the following three cases.

Case 1: When $n \equiv 0 (mod\ 3), n \geq 15$

The SVM labeling of graph obtained from C_{15} is shown in the following Figure 7.10.

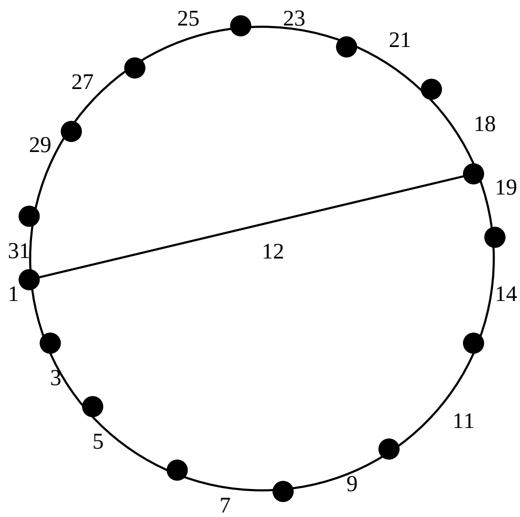


Figure 7.10: Super Vertex Mean Labeling of $C_{15} \cup \{uv\}$.

For $n \geq 18$ define $f : E(G) \to \{1, 2, 3, \cdots, 2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 2i-1, & \text{if } 1 \leq i \leq 6, \\ 2i, & \text{if } 7 \leq i \leq \frac{n}{3}+2, \\ 2i+1, & \text{if } \frac{n}{3}+3 \leq i \leq \frac{2}{3}n-3, \\ 2i+2, & \text{if } \frac{2}{3}n-2 \leq i \leq n-7, \\ 2i+1, & \text{if } n-6 \leq i \leq n, \\ 12, & \text{if } i = n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\frac{n}{3}+5, & \text{if } i=1, \\ 2i-2, & \text{if } 2 \le i \le 6, \\ 2i-1, & \text{if } 7 \le i \le \frac{n}{3}+2, \\ 2i, & \text{if } \frac{n}{3}+3 \le i \le \frac{2}{3}n-3, \\ 2i+1, & \text{if } \frac{2}{3}n-2 \ \le i \le n-7, \\ 4\frac{n}{3}-4, & \text{if } i=n-6, \\ 2i, & \text{if } n-5 \le i \le n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1,2,3,\cdots,2n+1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph.

Case 2: When $n \equiv 1(mod\,3), n \ge 16$

Define $f : E(G) \to \{1,2,3,\cdots,2n+1\}$ as follows,

$$f(e_i) = \begin{cases} 2i-1, & \text{if } 1 \le i \le 3, \\ 2i, & \text{if } 4 \le i \le \lceil\frac{n}{3}\rceil, \\ 2i+1, & \text{if } \lceil\frac{n}{3}\rceil+1 \ \le i \le 2\lceil\frac{n}{3}\rceil-5, \\ 2i+2, & \text{if } 2\lceil\frac{n}{3}\rceil-4 \ \le i \le n-7, \\ 2i+1, & \text{if } n-6 \le i \le n, \\ 6, & \text{if } i=n+1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 1, & \text{if } i = 1, \\ 2i - 2, & \text{if } 2 \leq i \leq 3, \\ 2i - 1, & \text{if } 4 \leq i \leq \lceil \frac{n}{3} \rceil, \\ 2i, & \text{if } \lceil \frac{n}{3} \rceil + 1 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 5, \\ 2i + 1, & \text{if } 2\lceil \frac{n}{3} \rceil - 4 \ \leq i \leq n - 7, \\ 4\lceil \frac{n}{3} \rceil - 8, & \text{if } i = n - 6, \\ 2i, & \text{if } n - 5 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n + 1\}$. Since f is a Super Vertex Mean labeling, G is a SVM - graph.

Case 3: When $n \equiv 2(mod\, 3), n \geq 14$

The SVM - labeling of graph obtained from C_{14} is shown in the following Figure 7.11.

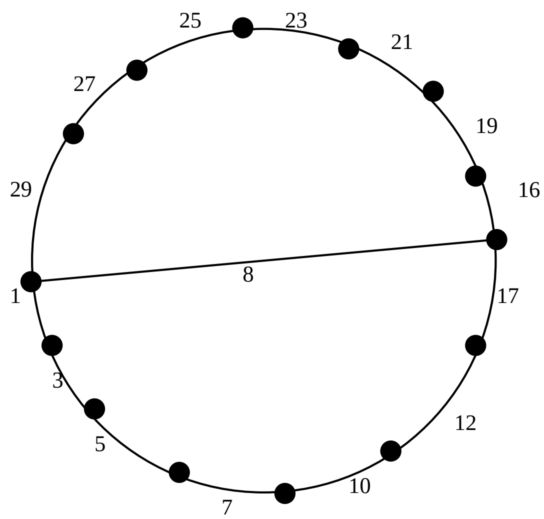


Figure 7.11: Super Vertex Mean Labeling of $C_{14} \cup \{uv\}$.

For $n \geq 17$ define $f : E(G) \to \{1, 2, 3, \cdots, 2n + 1\}$ as follows,

$$f(e_i) = \begin{cases} 2i - 1, & \text{if } 1 \leq i \leq 4, \\ 2i, & \text{if } 5 \leq i \leq \lceil \frac{n}{3} \rceil + 1, \\ 2i + 1, & \text{if } \lceil \frac{n}{3} \rceil + 2 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 4, \\ 2i + 2, & \text{if } 2\lceil \frac{n}{3} \rceil - 3 \ \leq i \leq n - 7, \\ 2i + 1, & \text{if } n - 6 \leq i \leq n, \\ 8, & \text{if } i = n + 1. \end{cases}$$

It can be easily verified that f is injective.

Then, the induced vertex labels are as follows:

$$f^v(u_i) = \begin{cases} 2\lceil \frac{n}{3} \rceil + 3, & \text{if } i = 1, \\ 2i - 2, & \text{if } 2 \leq i \leq 4, \\ 2i - 1, & \text{if } 5 \leq i \leq \lceil \frac{n}{3} \rceil + 1, \\ 2i, & \text{if } \lceil \frac{n}{3} \rceil + 2 \ \leq i \leq 2\lceil \frac{n}{3} \rceil - 4, \\ 2i + 1, & \text{if } 2\lceil \frac{n}{3} \rceil - 3 \ \leq i \leq n - 7, \\ 4\lceil \frac{n}{3} \rceil - 6, & \text{if } i = n - 6, \\ 2i, & \text{if } n - 5 \leq i \leq n. \end{cases}$$

Clearly it can be proved that the set of edge labels and the induced vertex labels is $\{1, 2, 3, \cdots, 2n + 1\}$. Since f is a Super Vertex Mean labeling, G is an SVM - graph. □

Conclusion and Future Direction

In Chapter 1, we have given the basic concepts in graph theory that are needed for the subsequent chapters. Also, we have presented certain graph labeling concepts that are used through the thesis in the second part of this chapter.

In Chapter 2, we have introduced Super Vertex Mean Labeling. We have proved that all cycles except C_4 are SVM graphs. Then we have discussed SVM labeling behavior with regard to type s-labeling of all cycles. Also, we have proved that ladder graph admits SVM labeling. Finally, we have discussed the SVM labeling behavior of all fan graphs.

In Chapter 3, we have proved that triangular snake, quadrilateral snakes, pentagonal snakes and hexagonal snakes are SVM graphs. Also we have proved that kC_n cyclic snake with k blocks of $C_n, n \geq 7$ and $n \equiv 3(mod\, 4)$, kC_n cyclic snake with k blocks of $C_n, n \geq 8$ and $n \equiv 0(mod\, 4)$, kC_n cyclic snake with k blocks of $C_n, n \geq 9$ and $n \equiv 1(mod\, 4)$ and kC_n cyclic snake with k blocks of $C_n, n \geq 10$ and $n \equiv 2(mod\, 4)$ are SVM graphs.

In Chapter 4, we have proved that linear quadrilateral snakes, linear pentagonal snakes, linear hexagonal snakes, linear heptagonal snakes admit SVM labeling. Then we have discussed SVM labeling behavior of kC_n cyclic snake with k, $k > 2$ blocks of $C_n, n \geq 8$ and $n \equiv 0(mod\, 2)$, $C_n, n \geq 9$ and $n \equiv 1(mod\, 4)$ and $C_n, n \geq 11$ and $n \equiv 3(mod\, 4)$.

In Chapter 5, we have proved that linear edge linked cyclic snakes $EL(kC_4)$, $EL(kC_5)$, $EL(kC_6)$, $EL(kC_7)$, $EL(kC_8)$, $EL(kC_9)$, $EL(kC_{10})$, $EL(kC_{11})$ are SVM graphs. Then we have discussed SVM labeling behavior of linear edge linked cyclic snakes $EL(kC_n)$, $n \equiv 0(mod\, 12)$ and $n \geq 12$, $n \equiv 1(mod\, 12)$ and $n \geq 13$, $n \equiv 2 \; or \; 8(mod\, 12)$ and $n \geq 14$, $n \equiv 3(mod\, 12)$ and $n \geq 15$, $n \equiv 4(mod\, 12)$ and $n \geq 16$, $n \equiv 5(mod\, 12)$ and $n \geq 17$, $n \equiv 6(mod\, 12)$ and $n \geq 18$, $n \equiv 7(mod\, 12)$ and $n \geq 19$, $n \equiv 9(mod\, 12)$ and $n \geq 21$,

$n \equiv 10(mod\, 12)$ and $n \geq 22$ and $n \equiv 11(mod\, 12)$ and $n \geq 23$.

In Chapter 6, we have proved that all the r-regular graphs of order ≤ 7 and all graphs having no isolated or pendant vertex and order ≤ 5, excluding C_4 are SVM-graphs.

In Chapter 7, we have proved that m copies of C_4 for $m \geq 2$, $C_m \cup C_n$ for $m, n \geq 3$, union of graphs, P_n^2 for $n \geq 3$ are SVM-graphs. Also we have proved that cycles with a chord connecting arc of distances 2, 3, 4, 5, 6, 7 admit SVM labeling.

We now point out some directions for interested researchers in the area of graph labeling:

1. Explore SVM labeling behaviour of other families of graphs that are not discussed here.

2. Explore SVM labeling behaviour of trees.

3. It will be an interesting problem if one discusses the SVM labeling behavior of non-linear edge linked cyclic snakes.

4. We can make an attempt to study SVM labeling behavior of splitting graph of cycle, shadow graph of cycle, middle graph of cycle, total graph of cycle.

5. We can make an attempt to study SVM labeling behavior of alternative triangular snake, alternative quadrilateral snakes etc.,

6. We can make an attempt to study SVM labeling behavior of cycle with a chord connecting arc of distance ≥ 8.

7. Do the r-regular graphs of order ≥ 7 admit SVM labeling?

8. Does square of cycle admit SVM labeling?

9. Does a kC- snake admit SVM labeling?

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Vitae

Mr. Sherry Geroge was born on 20th May 1976 at Kothamangalam, in Ernakulam District of Kerala. He completed his SSLC examination from St.Josephs School, Velielchal in 1991, and did his P.D.C. from St.Thomas College, Palai in 1993. From St.Josephs College, Bangalore he achieved his B.Sc. in Physics, Chemistry and Mathematics in the year 1997. He is awarded M.A. in Philosophy from Loyola College, Chennai. He also has completed his B.Ed. from Magadh University, Patna in 2005. Later in the year 2013, he completed Masters in Mathematics from St.Xaviers College (Autonomous), Palayamkottai of Manonmaniam University, Tirunelveli. He has published three research articles in national and international journals. He has sent another 5 research articles to various publications and waiting for communication from them. As a student of M.Sc. (Mathematics), he has participated in Paper Presentation Competitions and won prizes. He also has attended national and international seminars conducted on Discrete Mathematics and Graph Theory. Before joining for his Ph.D. programme in the year 2014, he has vast experience as a teacher of Mathematics in St.Xaviers School, Bhiwadi, Rajasthan, St.Xaviers School, Jaipur and St.Xaviers School, Delhi-54.

SHERRY GEORGE
RESEARCH SCHOLAR

Annexure

The following published papers and accepted paper are attached:

1. Lourdusamy, A, & Sherry George, 2017, 'Super Vertex Mean Graphs of Order ≤ 7', J. Appl. Math. & Informatics, Vol.35, No.5-6, PP.565-586.

2. Lourdusamy, A, Seenivasan, M, Sherry George & Revathy, R, 2014, 'Super Vertex-Mean Graphs', Sciencia Acta Xaveriana, Vol.5, No.2, PP.39-46.

3. Lourdusamy, A, & Sherry George, 2016, 'Super Vertex Mean Labeling of Cyclic Snakes', Information in Sciences and Computing, (Article ID ISC630416), Vol. 2016, PP.1-25.